The index quasi-periodicity and multiplicity of closed geodesics

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Abstract

In this paper, we prove the existence of at least two distinct closed geodesics on every compact simply connected irreversible or reversible Finsler (including Riemannian) manifold of dimension not less than 2.

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1 Introduction and main results

The closed geodesic problem is a traditional and active topic in dynamical systems and differential geometry for more than one hundred years. Studies of closed geodesics can be traced back to J. Jacobi, J. Hadamard, H. Poincaré, G. D. Birkhoff, M. Morse, L. Lyusternik and Schnirelmann and others. Specially G. D. Birkhoff established the existence of at least one closed geodesic on every Riemannian sphere S^d with $d \geq 2$ (cf. [Bir1]). Later L. Lyusternik and A. Fet proved the existence of at least one closed geodesic on every compact Riemannian manifold (cf. [LyF1]). Such a variational proof works also for Finlser metrics on compact manifolds and produces at least one closed geodesic on every such manifold. An important breakthrough on this study is due to V. Bangert [Ban2] and J. Franks [Fra1] around 1990, who proved the existence of infinitely many

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closed geodesics on every Riemannian 2-sphere (cf. also [Hin2] and [Hin3] for new proofs of some parts of this result).

For irreversible Finsler manifolds, the closed geodesic problem is more delicate as discovered by A. Katok via his famous example of 1973 which yields some irreversible Finsler metrics on S^d with precisely $2[(d+1)/2]$ distinct prime closed geodesics (cf. [Kat1] and [Zil1]). In [HWZ1] of 2003, H. Hofer, K. Wysocki and E. Zehnder proved that there exist either two or infinitely many distinct prime closed geodesics on a Finsler (S^2, F) provided that all the iterates of all closed geodesics are non-degenerate and the stable and unstable manifolds of all hyperbolic closed geodesics intersect transversally. In [BaL1] of 2005 published in 2010, V. Bangert and Y. Long proved that on every irreversible Finsler S^2 there exist always at least two distinct prime closed geodesics (cf. also [LoW2]).

Here recall that on a Finsler manifold (M, F), a closed geodesic $c : S^1 = \mathbf{R}/\mathbf{Z} \to M$ is *prime*, if it is not a multiple covering (i.e., iteration) of any other closed geodesic. Here the m-th iteration c^m of c is defined by $c^m(t) = c(mt)$ for $m \in \mathbf{N}$. The inverse curve c^{-1} of c is defined by $c^{-1}(t) = c(1-t)$ for $t \in S^1$. Two prime closed geodesics c_1 and c_2 on a Finsler manifold (M, F) (or Riemannian manifold (M, g)) are *distinct* (or *geometrically distinct*), if they do not differ by an S^1-action (or $O(2)$-action). We denote by $\mathrm{CG}(M, F)$ the set of all distinct closed geodesics on (M, F) for Finsler or Riemannian metric F on M.

A long-standing conjecture on the closed geodesics is

$$^{\#}\mathrm{CG}(M, g) = +\infty, \tag{1.1}$$

for every Riemannian metric g on any compact manifold M with $\dim M \geq 2$. Correspondingly for Finsler manifolds, it is conjectured (cf. [Lon6]) that for each positive integer n there exist positive integers $1 \leq p_n \leq q_n$ with $p_n \to +\infty$ as $n \to +\infty$ such that there holds

$$^{\#}\mathrm{CG}(M, F) \in [p_n,\, q_n] \cup \{+\infty\}, \tag{1.2}$$

for every Finsler metric F on each compact manifold M satisfying $\dim M = n$.

Note that by the results of [Ban2] and [Fra1] and the classification of 2-dimensional compact manifolds, the conjecture (1.1) was proved when $\dim M = 2$. Similarly by the results of [Kat1] and [BaL1], we have $p_2 = 2$.

In the study of the conjecture (1.1), D. Gromoll and W. Meyer [GrM1] in 1969 proved the following result:

Theorem A. ([GrM1]) *On a compact Riemannian manifold there exist infinitely many closed geodesics, if the free loop space of this manifold has an unbounded sequence of Betti numbers.*

Stimulated by this result, M. Vigué-Poirrier and D. Sullivan [ViS1] in 1976 proved:

Theorem B. ([ViS1]) *The free loop space of a compact simply connected Riemannian manifold M has no unbounded sequence of Betti numbers if and only if the rational cohomology algebra of M possess only one generator.*

Both of these two theorems were generalized to corresponding Finsler manifolds by H. Matthias in 1980 (cf. [Mat1]). Therefore based on these two theorems, the most interesting manifolds in this multiplicity problem are those compact simply connected manifolds satisfying

$$H^*(M;\mathbf{Q}) \cong T_{d,h+1}(x) = \mathbf{Q}[x]/(x^{h+1}=0) \tag{1.3}$$

with a generator x of degree $d \geq 2$ and hight $h+1 \geq 2$. The most important examples here are certainly spheres S^d of dimension d.

Besides these results, when the dimension of a compact simply connected manifold is greater than 2, we are not aware of any multiplicity results on the existence of at least two closed geodesics without pinching, generic, bumpy or other conditions even on spheres (cf. [Ano1], [Ban1], [Kli1], [BTZ1], [BTZ2], [DuL1], [DuL2], [LoW1], [Rad3], [Rad4], [Rad5], [Rad6]), except the Theorem C below proved recently in [LoD1] for the 3-dimensional case and [DuL3] for the 4-dimensional case.

Theorem C. ([LoD1], [DuL3]) *There exist always at least two distinct prime (geometrically distinct) closed geodesics for every irreversible (or reversible, specially Riemannian) Finsler metric on every* 3 *or* 4*-dimensional compact simply connected manifold.*

In this paper, we further generalize Theorem C to all compact simply connected Finsler as well as Riemannian manifolds and prove the following results.

Theorem 1.1. *For every irreversible Finsler metric F on any compact simply connected manifold of dimension at least* 2*, there exist always at least two distinct prime closed geodesics.*

Theorem 1.2. *For every reversible Finsler metric F on any compact simply connected manifold of dimension at least* 2*, there exist always at least two geometrically distinct closed geodesics. In particular, it holds for every such Riemannian manifold.*

Next we briefly describe the main ideas in the proofs of Theorems 1.1 and 1.2.

In [LoD1], we classified all the closed geodesics into *rational* and *irrational* two classes according to the basic normal form decomposition of their linearized Poincaré maps as symplectic matrices introduced in [Lon2] in 1999. Then in [LoD1], we established periodicity of the Morse indices and homological information of iterates of orientable rational closed geodesics on any Finsler manifold (M,F). Specially we proved

$$i(c^{n+m}) = i(c^n) + i(c^m) + \overline{p}(c), \qquad \nu(c^{n+m}) = \nu(c^m), \quad \forall\, m \in \mathbf{N}, \tag{1.4}$$

where $n = n(c)$ is the analytical period of a prime closed geodesic c, cf. (4.1) below, and $\overline{p}(c)$ is a constant depends only on the linearized Poincaré map P_c of c. We proved also a boundedness property of Morse indices in iterates of every prime orientable rational closed geodesic c:

$$i(c^m) + \nu(c^m) \leq i(c^n) + \overline{p}(c) + \dim M - 3, \quad \forall\, 1 \leq m \leq n-1. \tag{1.5}$$

If (M,F) is a compact simply connected Finsler manifold and possesses only one prime closed geodesic c, and if c is rational, based on the properties (1.4) and (1.5) we established in [LoD1] and [DuL3] the following identity

$$B(d,h)(i(c^n)+\overline{p}(c))+(-1)^{i(c^n)+\mu}\kappa=\sum_{j=\mu-\overline{p}(c)+1}^{i(c^n)+\mu}(-1)^j b_j, \tag{1.6}$$

for some integer $\kappa\geq 0$, where $\mu=\overline{p}(c)+\dim M-3$, and $B(d,h)$ depends only on d and h and is given in Lemma 2.4 below, b_js are Betti numbers of the relative free loop spaces defined in Lemmas 2.5 and 2.6 below. Then using (1.6), and our computations on the precise sum of Betti numbers, we obtain a contradiction and conclude that the only one prime closed geodesic c on M can not be rational.

Now in the current paper, our main idea is to generalize the above method on rational closed geodesics to every closed geodesic on compact simply connected manifolds. Suppose that there exists only one prime closed geodesic c on a compact simply connected Finsler manifold (M,F). The new observations in the current paper are the following:

(i) When c is irrational, suppose the basic normal form decomposition of the linearized Poincaré map P_c of c contains k irrational rotation matrices. In this case, we can not hope the periodicity (1.4) to be still true anymore for the analytical period $n=n(c)$ and the constant $\overline{p}(c)$. But using the mod one uniform distribution property of irrational numbers, we can still get a local version of (1.4), i.e., there exists a large enough even integer $T\in n(c)\mathbf{N}$ such that for some integer $m_0=m_0(c)>1$ depending on c only there holds

$$i(c^{T+m})=i(c^T)+i(c^m)+p(c),\quad \nu(c^{T+m})=\nu(c^m),\qquad \forall\, 1\leq m\leq m_0, \tag{1.7}$$

where $p(c)=\overline{p}(c)+2(A-k)$ for some integer $1\leq A\leq k$ depending on P_c only. We call such a property the *quasi-periodicity* of Morse indices of iterates c^m.

(ii) Similarly for irrational c, we can not hope (1.5) to be true for all multiples of n. But using estimates on Morse indices of iterates of irrational closed geodesics established in [DuL3], we can get also a similar version of (1.5), i.e., we can further choose the integer $T\in n(c)\mathbf{N}$ so that there holds

$$i(c^m)+\nu(c^m)\leq i(c^T)+p(c)+\dim M-3,\qquad \forall\, 1\leq m\leq T-1. \tag{1.8}$$

(iii) Now by computing out the alternating sum of the dimensions of all the critical modules of c^m with $1\leq m\leq T$, and then comparing with the Betti numbers of the free loop space pairs on M (i.e., b_js below), we obtain the following version of (1.6) which holds at the iteration T: i.e., there exists an integer $\kappa\geq 0$ such that

$$B(d,h)(i(c^T)+p(c))+(-1)^{i(c^T)+\mu}\kappa=\sum_{j=\mu-p(c)+1}^{i(c^T)+\mu}(-1)^j b_j, \tag{1.9}$$

where $\mu = p(c) + \dim M - 3$. Note that (1.7)-(1.9) are automatically reduced to (1.4)-(1.6) when c is rational.

(iv) Then the precise sum of Betti numbers on the right hand side of (1.9) yields a contradiction, and shows that there must exist at least two distinct closed geodesics.

Here we should point out that the identity (1.9) (or (1.6)) is rather different from the Morse inequalities, because the term $B(d,h)(i(c^T) + p(c))$ in (1.9) (or the corresponding term in (1.6)) represents the alternating sum of dimensions of all local critical modules of c^m with $1 \le m \le T$ (or $1 \le m \le n$), which is the alternating sum of all terms on or below the Tth (or nth) horizontal line in the Figure (5.54) below, and is not the alternating sum of Morse type numbers of c^ms with dimensions less than a fixed integer, which is the alternating sum of all terms on the left of some fixed vertical line in the Figure (5.54) below. In fact in our case, firstly the alternating sum of Morse type numbers with dimensions less than some fixed integer in the Morse inequality may not be computable, because in general there may not exist such a vertical line in the Figure (5.54) below such that all non-trivial critical modules of each iterate c^m appears only on one side of this vertical line. Secondly, even if it is computable, it is still not clear whether the corresponding Morse inequalities may yield any contradiction.

Note that in his famous book [Mor1], M. Morse proved that for any given integer $N > 0$ the global homology of a d-dimensional ellipsoid $\mathcal{E}_d$ at all dimensions less than N can be produced by iterates of the $(d+1)$ main ellipses only, provided $\mathcal{E}_d$ is sufficiently close to the ball and all of its semi-axis are different. His this example explains why the iterate T in our proof should be sufficiently large and carefully chosen.

For reader's conveniences, in Section 2 we briefly review some known results on closed geodesics and Betti numbers of the S^1-invariant free loop space of compact simply connected manifolds satisfying the condition (1.3). In Section 3 we briefly review basic normal form decompositions of symplectic matrices and the precise index iteration formulae of symplectic paths established by Y. Long in [Lon2] and [Lon3] together with the orientability of closed geodesics. In Section 4, we establish the quasi-periodicity (1.7) and the boundedness estimate (1.8) of iterated indices of closed geodesics. In Section 5, using the index quasi-periodicity we prove some homological isomorphism theorems of energy critical level pairs when there exists only one prime closed geodesic, and then establish the identity (1.9). In Section 6 we give proofs of Theorems 1.1 and 1.2.

In this paper, we denote by $\mathbf{N}$, $\mathbf{N}_0$, $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{C}$ the sets of positive integers, non-negative integers, integers, rational numbers, real numbers, and complex numbers respectively. We define the functions $[a] = \max\{k \in \mathbf{Z} \,|\, k \le a\}$, $\{a\} = a - [a]$, $E(a) = \min\{k \in \mathbf{Z} \,|\, k \ge a\}$ and $\varphi(a) = E(a) - [a]$. Denote by $^{\#}A$ the number of elements in a finite set A. When S^1 acts on a topological space X, we denote by $\overline{X}$ the quotient space X/S^1. In this paper, we use only singular homology modules with $\mathbf{Q}$-coefficients.

2 Critical point theory of closed geodesics

2.1 Critical modules for closed geodesics

Let M be a manifold with a Finsler metric F. Closed geodesics are critical points of the energy functional $E(\gamma) = \frac{1}{2}\int_{S^1} F(\gamma(t), \dot{\gamma}(t))^2 dt$ on the Hilbert manifold ΛM of H^1-maps from S^1 to M. An S^1-action is defined by $(s \cdot \gamma)(t) = \gamma(t+s)$ for all $\gamma \in \Lambda M$ and $s, t \in S^1$. The index form of the functional E is well defined along any closed geodesic c on M, which we denote by $E''(c)$. As usual, denote by $i(c)$ and $\nu(c)$ the Morse index and nullity of E at c. For a closed geodesic c, denote by c^m the m-fold iteration of c and $\Lambda(c^m) = \{\gamma \in \Lambda M \mid E(\gamma) < E(c^m)\}$. Recall that respectively the *mean index* $\hat{i}(c)$ and the S^1-*critical modules* of c^m are defined by

$$\hat{i}(c) = \lim_{m\to\infty} \frac{i(c^m)}{m}, \quad \overline{C}_*(E, c^m) = H_*\left((\Lambda(c^m) \cup S^1 \cdot c^m)/S^1, \Lambda(c^m)/S^1\right). \tag{2.1}$$

If c has multiplicity m, then the subgroup $\mathbf{Z}_m = \{\frac{n}{m} : 0 \le n < m\}$ of S^1 acts on $\overline{C}_k(E, c)$. As on page 59 of [Rad2], for $m \ge 1$, let $H_*(X, A)^{\pm\mathbf{Z}_m} = \{[\xi] \in H_*(X, A) : T_*[\xi] = \pm\xi\}$, where T is a generator of the $\mathbf{Z}_m$ action. On S^1-critical modules of c^m, the following lemma holds:

Lemma 2.1. (cf. [Rad2], [BaL1], [LoD1]) *Suppose c is a prime closed geodesic on a Finsler manifold M. Then there exist two sets $U^-_{c^m}$ and N_{c^m}, the so-called local negative disk and the local characteristic manifold at c^m respectively, such that $\nu(c^m) = \dim N_{c^m}$ and*

$$\begin{aligned} \overline{C}_q(E, c^m) &\equiv H_q\left((\Lambda(c^m) \cup S^1 \cdot c^m)/S^1, \Lambda(c^m)/S^1\right) \\ &= \left(H_{i(c^m)}(U^-_{c^m} \cup \{c^m\}, U^-_{c^m}) \otimes H_{q-i(c^m)}(N^-_{c^m} \cup \{c^m\}, N^-_{c^m})\right)^{+\mathbf{Z}_m}, \end{aligned}$$

(i) When $\nu(c^m) = 0$, there holds

$$\overline{C}_q(E, c^m) = \begin{cases} \mathbf{Q}, & \text{if } \ i(c^m) = i(c)\,(\mathrm{mod} 2) \ \text{ and } \ q = i(c^m), \\ 0, & \text{otherwise}, \end{cases}$$

(ii) When $\nu(c^m) > 0$, let $\epsilon(c^m) = (-1)^{i(c^m)-i(c)}$, then there holds

$$\overline{C}_q(E, c^m) = H_{q-i(c^m)}(N^-_{c^m} \cup \{c^m\}, N^-_{c^m})^{\epsilon(c^m)\mathbf{Z}_m}.$$

Let

$$k_j(c^m) \equiv \dim\, H_j(N^-_{c^m} \cup \{c^m\}, N^-_{c^m}), \quad k_j^{\pm 1}(c^m) \equiv \dim\, H_j(N^-_{c^m} \cup \{c^m\}, N^-_{c^m})^{\pm\mathbf{Z}_m}. \tag{2.2}$$

Then we have

Lemma 2.2. (cf. [Rad2], [BaL1], [LoD1]) *Let c be a closed geodesic on a Finsler manifold M.*

(i) There hold $0 \le k_j^{\pm 1}(c^m) \le k_j(c^m)$ for $m \ge 1$ and $j \in \mathbf{Z}$, $k_j(c^m) = 0$ whenever $j \notin [0, \nu(c^m)]$ and $k_0(c^m) + k_{\nu(c^m)}(c^m) \le 1$. If $k_0(c^m) + k_{\nu(c^m)}(c^m) = 1$, then $k_j(c^m) = 0$ when $j \in (0, \nu(c^m))$.

(ii) For any $m \in \mathbf{N}$, there hold $k_0^{+1}(c^m) = k_0(c^m)$ and $k_0^{-1}(c^m) = 0$. In particular, if c^m is non-degenerate, there hold $k_0^{+1}(c^m) = k_0(c^m) = 1$, and $k_0^{-1}(c^m) = k_j^{\pm 1}(c^m) = 0$ for all $j \neq 0$.

(iii) Suppose for some integer $m = np \geq 2$ with n and $p \in \mathbf{N}$ the nullities satisfy $\nu(c^m) = \nu(c^n)$. Then there hold $k_j(c^m) = k_j(c^n)$ and $k_j^{\pm 1}(c^m) = k_j^{\pm 1}(c^n)$ for any integer j.

Let (M, F) be a compact and simply connected Finsler manifold with finitely many prime closed geodesics. It is well known that for every prime closed geodesic c on (M, F), there holds either $\hat{i}(c) > 0$ and then $i(c^m) \to +\infty$ as $m \to +\infty$, or $\hat{i}(c) = 0$ and then $i(c^m) = 0$ for all $m \in \mathbf{N}$. Denote those prime closed geodesics on (M, F) with positive mean indices by $\{c_j\}_{1 \leq j \leq k}$. In [Rad1] and [Rad2], Rademacher established a celebrated mean index identity relating all the c_js with the global homology of M (cf. Section 7, specially Satz 7.9 of [Rad2]) for compact simply connected Finsler manifolds. A refined version of this identity with precise coefficients was proved in [BaL1], [LoW1], and [LoD1].

For each $m \in \mathbf{N}$, let $\epsilon = \epsilon(c^m) = (-1)^{i(c^m) - i(c)}$ and

$$\begin{aligned} K(c^m) &\equiv (k_0^{\epsilon}(c^m), k_1^{\epsilon}(c^m), \ldots, k_{2\dim M - 2}^{\epsilon}(c^m)) \\ &= (k_0^{\epsilon(c^m)}(c^m), k_1^{\epsilon(c^m)}(c^m), \ldots, k_{\nu(c^m)}^{\epsilon(c^m)}(c^m), 0, \ldots, 0). \end{aligned} \tag{2.3}$$

Lemma 2.3. (cf. Lemmas 7.1 and 7.2 of [Rad2], cf. also [LoD1]) *Let c be a prime closed geodesic on a compact Finsler manifold (M, F). Then there exists a minimal integer $N = N(c) \in \mathbf{N}$ such that $\nu(c^{m+N}) = \nu(c^m)$, $i(c^{m+N}) - i(c^m) \in 2\mathbf{Z}$, and $K(c^{m+N}) = K(c^m)$ for all $m \in \mathbf{N}$.*

Lemma 2.4. (cf. [Rad2], [BaL1], [LoW1], [LoD1]) *Let (M, F) be a compact simply connected Finsler manifold with $H^*(M, \mathbf{Q}) = T_{d,h+1}(x)$ for some integers $d \geq 2$ and $h \geq 1$. Denote prime closed geodesics on (M, F) with positive mean indices by $\{c_j\}_{1 \leq j \leq k}$ for some $k \in \mathbf{N}$. Then the following identity holds*

$$\sum_{j=1}^{k} \frac{\hat{\chi}(c_j)}{\hat{i}(c_j)} = B(d, h) = \begin{cases} -\frac{h(h+1)d}{2d(h+1)-4}, & d \text{ even}, \\ \frac{d+1}{2d-2}, & d \text{ odd}, \end{cases} \tag{2.4}$$

where $\dim M = hd$, $h = 1$ when M is a sphere S^d of dimension d and

$$\hat{\chi}(c) = \frac{1}{N(c)} \sum_{\substack{0 \leq l_m \leq \nu(c^m) \\ 1 \leq m \leq N(c)}} (-1)^{i(c^m) + l_m} k_{l_m}^{\epsilon(c^m)}(c^m) \ \in \ \mathbf{Q}. \tag{2.5}$$

2.2 The structure of $H_*(\Lambda M/S^1, \Lambda^0 M/S^1; \mathbf{Q})$

Set $\overline{\Lambda}^0 = \overline{\Lambda}^0 M = \{\text{constant point curves in } M\} \cong M$. Let (X, Y) be a space pair such that the Betti numbers $b_i = b_i(X, Y) = \dim H_i(X, Y; \mathbf{Q})$ are finite for all $i \in \mathbf{Z}$. As usual the *Poincaré series* of (X, Y) is defined by the formal power series $P(X, Y) = \sum_{i=0}^{\infty} b_i t^i$. We need the following

version of results on Betti numbers. The precise computations on each Betti number in Lemma 2.6 and sums of Betti numbers in Lemmas 2.5 and 2.6 were given in [LoD1] and [DuL3].

Lemma 2.5. (cf. Theorem 2.4 and Remark 2.5 of [Rad1], Proposition 2.4 of [LoD1], Lemma 2.5 of [DuL3]) *Let (S^d, F) be a d-dimensional Finsler sphere.*

(i) *When d is odd, the Betti numbers are given by*

$$\begin{aligned} b_j &= \mathrm{rank} H_j(\Lambda S^d/S^1, \Lambda^0 S^d/S^1; \mathbf{Q}) \\ &= \begin{cases} 2, & \text{if} \quad j \in \mathcal{K} \equiv \{k(d-1) \,|\, 2 \le k \in \mathbf{N}\}, \\ 1, & \text{if} \quad j \in \{d-1+2k \,|\, k \in \mathbf{N}_0\} \setminus \mathcal{K}, \\ 0 & \text{otherwise}. \end{cases} \end{aligned} \tag{2.6}$$

For any $k \in \mathbf{N}$ and $k \ge d-1$, there holds

$$\begin{aligned} \sum_{j=0}^{k} (-1)^j b_j &= \sum_{0 \le 2j \le k} b_{2j} \\ &= \frac{k(d+1)}{2(d-1)} - \frac{d-1}{2} - \epsilon_{d,1}(k) \\ &\le \frac{k(d+1)}{2(d-1)} - \frac{d-1}{2}. \end{aligned} \tag{2.7}$$

where $\epsilon_{d,1}(k) = \{\frac{k}{d-1}\} + \{\frac{k}{2}\} \in [0, \frac{3}{2} - \frac{1}{2(d-1)})$.

(ii) *When d is even, the Betti numbers are given by*

$$\begin{aligned} b_j &= \mathrm{rank} H_j(\Lambda S^d/S^1, \Lambda^0 S^d/S^1; \mathbf{Q}) \\ &= \begin{cases} 2, & \text{if} \quad j \in \mathcal{K} \equiv \{k(d-1) \,|\, 3 \le k \in (2\mathbf{N}+1)\}, \\ 1, & \text{if} \quad j \in \{d-1+2k \,|\, k \in \mathbf{N}_0\} \setminus \mathcal{K}, \\ 0 & \text{otherwise}. \end{cases} \end{aligned} \tag{2.8}$$

For any $k \in \mathbf{N}$ and $k \ge d-1$, there holds

$$-\sum_{j=0}^{k} (-1)^j b_j = \sum_{0 \le 2j-1 \le k} b_{2j-1} \le \frac{kd}{2(d-1)} - \frac{d-2}{2}. \tag{2.9}$$

For a compact and simply connected Finsler manifold M with $H^*(M; \mathbf{Q}) \cong T_{d,h+1}(x)$, when d is odd, then $x^2 = 0$ and $h = 1$ in $T_{d,h+1}(x)$. Thus M is rationally homotopy equivalent to S^d (cf. Remark 2.5 of [Rad1]). Therefore, next we only consider the case when d is even.

Lemma 2.6. (cf. Theorem 2.4 of [Rad1], Lemma 2.6 of [DuL3]) *Let M be a compact simply connected manifold with $H^*(M; \mathbf{Q}) \cong T_{d,h+1}(x)$ for some integer $h \ge 1$ and even integer $d \ge 2$. Let $D = d(h+1) - 2$ and*

$$\begin{aligned} \Omega(d,h) = \{k \in 2\mathbf{N}-1 \quad | \quad & iD \le k-(d-1) = iD + jd \le iD + (h-1)d \\ & \text{for some } i \in \mathbf{N} \text{ and } j \in [1, h-1]\}. \end{aligned} \tag{2.10}$$

Then the Betti numbers of the free loop space of M defined by $b_q = \mathrm{rank} H_q(\Lambda M/S^1, \Lambda^0 M/S^1; \mathbf{Q})$ for $q \in \mathbf{Z}$ are given by

$$b_q = \begin{cases} 0, & \text{if } q \text{ is even or } q \le d-2, \\ [\frac{q-(d-1)}{d}]+1, & \text{if } q \in 2\mathbf{N}-1 \text{ and } d-1 \le q < d-1+(h-1)d, \\ h+1, & \text{if } q \in \Omega(d,h), \\ h, & \text{otherwise.} \end{cases} \tag{2.11}$$

For every integer $k \ge d-1+(h-1)d = hd-1$, we have

$$\begin{aligned} \sum_{q=0}^{k} b_q &= \frac{h(h+1)d}{2D}(k-(d-1)) - \frac{h(h-1)d}{4} + 1 + \epsilon_{d,h}(k) \\ &< h(\frac{D}{2}+1)\frac{k-(d-1)}{D} - \frac{h(h-1)d}{4} + 2, \end{aligned} \tag{2.12}$$

where

$$\begin{aligned} \epsilon_{d,h}(k) &= \{\frac{D}{hd}\{\frac{k-(d-1)}{D}\}\} - (\frac{2}{d}+\frac{d-2}{hd})\{\frac{k-(d-1)}{D}\} \\ &\quad -h\{\frac{D}{2}\{\frac{k-(d-1)}{D}\}\} - \{\frac{D}{d}\{\frac{k-(d-1)}{D}\}\}, \end{aligned} \tag{2.13}$$

and there hold $\epsilon_{d,h}(k) \in (-(h+2),1)$ and $\epsilon_{d,1}(k) \in (-2,0]$ for all integer $k \ge d-1$.

3 A review of the precise index iteration formulae for symplectic paths

For $d \in \mathbf{N}$ and $\tau > 0$, denote by $\mathrm{Sp}(2d)$ the symplectic group whose elements are $2d \times 2d$ real symplectic matrices and let

$$\mathcal{P}_\tau(2d) = \{\gamma \in C([0,\tau], \mathrm{Sp}(2d)) \,|\, \gamma(0) = I\}.$$

An index function theory $(i_\omega(\gamma), \nu_\omega(\gamma))$ for every symplectic path $\gamma \in \mathcal{P}_\tau(2d)$ parametrized by $\omega \in \mathbf{U} = \{z \in \mathbf{C} \,|, |z| = 1\}$ was introduced by Y. Long in [Lon2] of 1999. This index function theory is based on the Maslov-type index theory $(i_1(\gamma), \nu_1(\gamma))$ for symplectic paths in $\mathcal{P}_\tau(2d)$ established by C. Conley, E. Zehnder in [CoZ1] of 1984, Y. Long and E. Zehnder in [LZe1] of 1990, and Y. Long in [Lon1] of 1990 (cf. [Lon5]). In [Lon2], Y. Long established also the basic normal form decomposition of symplectic matrices. Based on this result he further established the precise iteration formulae of indices of symplectic paths in [Lon3] of 2000. These results form the basis of our study on the Morse indices and homological properties of iterates of closed geodesics. Here we briefly review these results.

As in [Lon5], denote by

$$N_1(\lambda, a) = \begin{pmatrix} \lambda & a \\ 0 & \lambda \end{pmatrix}, \qquad \text{for } \lambda = \pm 1,\ a \in \mathbf{R}, \tag{3.1}$$

$$H(b) = \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}, \qquad \text{for } b \in \mathbf{R} \setminus \{0, \pm 1\}, \tag{3.2}$$

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad \text{for } \theta \in (0, \pi) \cup (\pi, 2\pi), \tag{3.3}$$

$$N_2(e^{\theta\sqrt{-1}}, B) = \begin{pmatrix} R(\theta) & B \\ 0 & R(\theta) \end{pmatrix}, \qquad \text{for } \theta \in (0, \pi) \cup (\pi, 2\pi) \text{ and}$$

$$B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \quad \text{with } b_j \in \mathbf{R}, \text{ and } b_2 \neq b_3. \tag{3.4}$$

Here $N_2(e^{\theta\sqrt{-1}}, B)$ is non-trivial if $(b_2 - b_3)\sin\theta < 0$, and trivial if $(b_2 - b_3)\sin\theta > 0$. In [Lon2]-[Lon4], these matrices are called *basic normal forms* of symplectic matrices.

As in [Lon5], given any two real matrices of the square block form

$$M_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}_{2i \times 2i}, \qquad M_2 = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}_{2j \times 2j},$$

the $\diamond$-sum (direct sum) of M_1 and M_2 is defined by the $2(i+j) \times 2(i+j)$ matrix

$$M_1 \diamond M_2 = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix}.$$

Definition 3.1. (cf. [Lon3] and [Lon5]) *For every $P \in \mathrm{Sp}(2d)$, the homotopy set $\Omega(P)$ of P in $\mathrm{Sp}(2d)$ is defined by*

$$\Omega(P) = \{N \in \mathrm{Sp}(2d) \,|\, \sigma(N) \cap \mathbf{U} = \sigma(P) \cap \mathbf{U} \equiv \Gamma \text{ and } \nu_\omega(N) = \nu_\omega(P)\, \forall \omega \in \Gamma\},$$

where $\sigma(P)$ denotes the spectrum of P, $\nu_\omega(P) \equiv \dim_{\mathbf{C}} \ker_{\mathbf{C}}(P - \omega I)$ for all $\omega \in \mathbf{U}$. The homotopy component $\Omega^0(P)$ of P in $\mathrm{Sp}(2d)$ is defined by the path connected component of $\Omega(P)$ containing P (cf. page 38 of [Lon5]).

Note that $\Omega^0(P)$ defines an equivalent relation among symplectic matrices. Specially we call two matrices N and $P \in \mathrm{Sp}(2d)$ *homotopic*, if $N \in \Omega^0(P)$, and in this case we write $N \approx P$.

Then the following decomposition theorem is proved in [Lon2] and [Lon3]

Theorem 3.2. (cf. Theorem 7.8 of [Lon2], Lemma 2.3.5 and Theorem 1.8.10 of [Lon5]) *For every $P \in \mathrm{Sp}(2d)$, there exists a continuous path $f \in \Omega^0(P)$ such that $f(0) = P$ and*

$$f(1) = N_1(1,1)^{\diamond p_-} \diamond I_{2p_0} \diamond N_1(1,-1)^{\diamond p_+}$$

$$
\begin{aligned}
&\diamond N_1(-1,1)^{\diamond q_-}\diamond(-I_{2q_0})\diamond N_1(-1,-1)^{\diamond q_+}\\
&\diamond R(\theta_1)\diamond\cdots\diamond R(\theta_k)\diamond R(\theta_{k+1})\diamond\cdots\diamond R(\theta_r)\\
&\diamond N_2(e^{\alpha_1\sqrt{-1}},A_1)\diamond\cdots\diamond N_2(e^{\alpha_{k_*}\sqrt{-1}},A_{k_*})\\
&\qquad\diamond N_2(e^{\alpha_{k_*+1}\sqrt{-1}},A_{k_*+1})\diamond\cdots\diamond N_2(e^{\alpha_{r_*}\sqrt{-1}},A_{r_*})\\
&\diamond N_2(e^{\beta_1\sqrt{-1}},B_1)\diamond\cdots\diamond N_2(e^{\beta_{k_0}\sqrt{-1}},B_{k_0})\\
&\qquad\diamond N_2(e^{\beta_{k_0+1}\sqrt{-1}},B_{k_0+1})\diamond\cdots\diamond N_2(e^{\beta_{r_0}\sqrt{-1}},B_{r_0})\\
&\diamond H(2)^{\diamond h_+}\diamond H(-2)^{\diamond h_-},
\end{aligned}
\tag{3.5}
$$

where $\frac{\theta_j}{2\pi}\notin\mathbf{Q}$ *for* $1\le j\le k$ *and* $\frac{\theta_j}{2\pi}\in\mathbf{Q}$ *for* $k+1\le j\le r$*;* $N_2(e^{\alpha_j\sqrt{-1}},A_j)$*'s are nontrivial basic normal forms with* $\frac{\alpha_j}{2\pi}\notin\mathbf{Q}$ *for* $1\le j\le k_*$ *and* $\frac{\alpha_j}{2\pi}\in\mathbf{Q}$ *for* $k_*+1\le j\le r_*$*;* $N_2(e^{\beta_j\sqrt{-1}},B_j)$*'s are trivial basic normal forms with* $\frac{\beta_j}{2\pi}\notin\mathbf{Q}$ *for* $1\le j\le k_0$ *and* $\frac{\beta_j}{2\pi}\in\mathbf{Q}$ *for* $k_0+1\le j\le r_0$*;* $p_-=p_-(P)$*,* $p_0=p_0(P)$*,* $p_+=p_+(P)$*,* $q_-=q_-(P)$*,* $q_0=q_0(P)$*,* $q_+=q_+(P)$*,* $r=r(P)$*,* $k=k(P)$*,* $r_j=r_j(P)$*,* $k_j=k_j(P)$ *with* $j=*,0$ *and* $h_+=h_+(P)$ *are nonnegative integers, and* $h_-=h_-(P)\in\{0,1\}$*;* θ_j*,* α_j*,* $\beta_j\in(0,\pi)\cup(\pi,2\pi)$*; these integers and real numbers are uniquely determined by* P *and satisfy*

$$
p_-+p_0+p_++q_-+q_0+q_++r+2r_*+2r_0+h_-+h_+=d. \tag{3.6}
$$

Based on Theorem 3.2, the homotopy invariance and symplectic additivity of the indices, the following precise iteration formula was proved in [Lon3]:

Theorem 3.3. (cf. [Lon3], Theorem 8.3.1 and Corollary 8.3.2 of [Lon5]) *Let* $\gamma\in\mathcal{P}_\tau(2d)$*. Denote the basic normal form decomposition of* $P\equiv\gamma(\tau)$ *by (3.5). Then we have*

$$
\begin{aligned}
i_1(\gamma^m) &= m(i_1(\gamma)+p_-+p_0-r)+2\sum_{j=1}^{r}E\left(\frac{m\theta_j}{2\pi}\right)-r\\
&\quad -p_--p_0-\frac{1+(-1)^m}{2}(q_0+q_+)\\
&\quad +2\sum_{j=k_*+1}^{r_*}\varphi\left(\frac{m\alpha_j}{2\pi}\right)-2(r_*-k_*), \qquad (3.7)\\
\nu_1(\gamma^m) &= \nu_1(\gamma)+\frac{1+(-1)^m}{2}(q_-+2q_0+q_+)+2\varsigma(m,\gamma(\tau)), \qquad (3.8)\\
\hat{i}(\gamma) &= i_1(\gamma)+p_-+p_0-r+\sum_{j=1}^{r}\frac{\theta_j}{\pi}, \qquad (3.9)
\end{aligned}
$$

where we denote by

$$
\begin{aligned}
\varsigma(m,\gamma(\tau)) &= (r-k)-\sum_{j=k+1}^{r}\varphi\left(\frac{m\theta_j}{2\pi}\right)\\
&\quad +(r_*-k_*)-\sum_{j=k_*+1}^{r_*}\varphi\left(\frac{m\alpha_j}{2\pi}\right)+(r_0-k_0)-\sum_{j=k_0+1}^{r_0}\varphi\left(\frac{m\beta_j}{2\pi}\right).
\end{aligned}
\tag{3.10}
$$

Let

$$\mathcal{M} \equiv \{N_1(1,b_1),\, b_1=0,1;\;\; N_1(-1,b_2),\, b_2=0,\pm 1;\;\; R(\theta),\, \theta\in(0,\pi)\cup(\pi,2\pi);\; H(-2)\}. \tag{3.11}$$

By Theorems 8.1.4-8.1.7 and 8.2.1-8.2.4 on pp179-187 of [Lon5], we have specially

Proposition 3.4. *Every path $\gamma\in\mathcal{P}_\tau(2)$ with end matrix homotopic to some matrix in $\mathcal{M}$ must have odd index $i_1(\gamma)$. Paths $\xi\in\mathcal{P}_\tau(2)$ ending at $N_1(1,-1)$ or $H(2)$ and $\eta\in\mathcal{P}_\tau(4)$ with end matrix homotopic to $N_2(\omega,B)$ must have even indices $i_1(\xi)$ and $i_1(\eta)$.*

The relation between the Morse indices of closed geodesics on Finsler manifolds and the above Maslov-type index theory for symplectic paths was studied by C. Liu and Y. Long in [LLo1] and C. Liu in [Liu1]. Specially we have

Proposition 3.5. (Theorem 1.1 and Remark 4.2 of [Liu1], cf. also Theorem 1.1 of [LLo1],) *For any closed geodesic c on a Finsler manifold (M,F) with $d=\dim M<+\infty$, denote its linearized Poincaré map by P_c. Then there exists a path $\gamma\in C([0,1],\mathrm{Sp}(2d-2))$ satisfying $\gamma(0)=I$, $\gamma(1)=P_c$, and*

$$\begin{aligned}(i(c),\nu(c)) &= (i_1(\gamma),\nu_1(\gamma)), \qquad \text{if } c \text{ is orientable}, &(3.12)\\ (i(c),\nu(c)) &= (i_{-1}(\gamma),\nu_{-1}(\gamma)), \qquad \text{if } c \text{ is unorientable and } d \text{ is even}. &(3.13)\end{aligned}$$

By this result, the above index iteration formulae (Theorem 3.3) can be applied to every orientable closed geodesic on Finsler and Riemannian manifolds. For unorientable closed geodesics, one can get a similar iteration formulae using results in [Lon5].

Remark 3.6. Note that every closed geodesic c on a simply connected Finsler manifold is always orientable and thus Theorem 3.3 can be applied to get $i(c^m)$ directly (cf. Section 2.1-Appendix on pages 136-141 of [Kli1]). In this paper we are interested in orientable closed geodesics.

Next we need the following results from [DuL3].

Proposition 3.7. (Corollary 3.19 of [DuL3]) *Let $v=(v_1,\ldots,v_k)\in(\mathbf{R}\setminus\mathbf{Q})^k$. Then there exists an integer A satisfying $[(k+1)/2]\le A\le k$ and a subset P of $\{1,\ldots,k\}$ containing A integers, such that for any integer $n\in\mathbf{N}$ and any small $\epsilon>0$ there exist infinitely many even integers T_1 and $T_2\in n\mathbf{N}$ satisfying respectively*

$$\begin{cases}\{T_1v_i\}>1-\epsilon, & \text{for } i\in P,\\ \{T_1v_j\}<\epsilon, & \text{for } j\in\{1,\ldots,k\}\setminus P,\end{cases} \tag{3.14}$$

$$\text{or}\quad \begin{cases}\{T_2v_i\}<\epsilon, & \text{for } i\in P,\\ \{T_2v_j\}>1-\epsilon, & \text{for } j\in\{1,\ldots,k\}\setminus P.\end{cases} \tag{3.15}$$

Theorem 3.8. (Theorem 3.21 of [DuL3]) (**Quasi-monotonicity of index growth for closed geodesics**) *Let c be an orientable closed geodesic with mean index $\hat{i}(c)>0$ on a Finsler manifold*

(M,F) of dimension $d\ge 2$. Denote the basic normal form decomposition of the linearized Poincaré map P_c of c by (3.5). Then there exist an integer A with $[(k+1)/2]\le A\le k$ and a subset P of integers $\{1,\ldots,k\}$ with A integers such that for any $\epsilon\in(0,1/4)$ there exist infinitely many sufficiently large even integer $T\in n\mathbf{N}$ satisfying

$$\left\{\frac{T\theta_j}{2\pi}\right\} \;>\; 1-\epsilon, \qquad \text{for } j\in P, \tag{3.16}$$

$$\left\{\frac{T\theta_j}{2\pi}\right\} \;<\; \epsilon, \qquad \text{for } j\in\{1,\ldots,k\}\setminus P. \tag{3.17}$$

Consequently we have

$$i(c^m)-i(c^T) \;\ge\; K_1\equiv \lambda+(q_0+q_+)+2(r-k)+2(r_*-k_*)+2A, \quad \forall\, m\ge T+1, \tag{3.18}$$

$$i(c^T)-i(c^m) \;\ge\; K_2\equiv \lambda-(q_0+q_+)+2k-2(r_*-k_*)-2A, \quad \forall\, 1\le m\le T-1, \tag{3.19}$$

where $\lambda=i(c)+p_-+p_0-r$, the integers p_-, p_0, q_0, q_+, r, k, r_* and k_* are defined in (3.5).

4 Properties of Morse indices of iterates of closed geodesics

Let (M,F) be a Finsler manifold of dimension d. As in [LoD1], a matrix $P\in\mathrm{Sp}(2d-2)$ is *rational*, if no basic normal form in (3.5) of P is of the form $R(\theta)$ with $\theta/\pi\in\mathbf{R}\setminus\mathbf{Q}$, and is *irrational*, otherwise. Let c be a closed geodesic on (M,F) whose linearized Poincaré map is denoted by P_c and then $P_c\in\mathrm{Sp}(2d-2)$. The closed geodesic c is *rational*, *irrational*, if so is P_c. The *analytical period* $n(c)$ of c is defined by

$$n(c)=\min\{j\in\mathbf{N}\,|\,\nu(c^j)=\max_{m\ge1}\nu(c^m),\;\; i(c^{m+j})-i(c^m)\in 2\mathbf{Z},\;\;\forall\, m\in\mathbf{N}\}. \tag{4.1}$$

One of the most important properties of $n=n(c)$ is

$$n(c)=N(c), \tag{4.2}$$

where $N(c)$ is defined by Lemma 2.3. This was proved by Lemma 3.10 of [DuL3].

Next we need

Definition 4.1. *For any closed geodesic c with $\hat{i}(c)>0$ on a Finsler manifold (M,F) of dimension d, Denote the basic normal form decomposition of the linearized Poincaré map P_c of c by (3.5). We define $m_0=m_0(c)$ as follows*

$$m_0(c)=\min\{m\in\mathbf{N}\,|\,i(c^{j+m})\ge d+4k,\,\forall\, j\ge 1\}.$$

Here k is defined in (3.5). Note that $\hat{i}(c)>0$ implies that $i(c^m)\to+\infty$ as $m\to+\infty$. Thus the integer m_0 is well-defined.

Lemma 4.2. *Let c be an orientable prime closed geodesic on a Finsler manifold $M = (M, F)$ with $\dim M < +\infty$. Then for every $m \in \mathbf{N}$, we have*

$$i(c^{2m}) = i(c^2) \mod 2, \quad i(c^{2m+1}) = i(c) \mod 2. \tag{4.3}$$

For any two positive integers $q|p$, we have

$$i(c^p) \geq i(c^q) \quad \text{and} \quad \nu(c^p) \geq \nu(c^q). \tag{4.4}$$

Proof. (4.3) follows from (3.7) of Theorem 3.3 immediately. The two inequalities in (4.4) follow from the Bott formulae (cf. Theorem 1 and its corollary on pages 177-178 of [Bot1]) immediately. In fact using notations of [Lon5] we have

$$i(c^p) = \sum_{\omega^p=1} i_\omega(c) = \sum_{\omega^q=1} i_\omega(c) + \sum_{\omega^p=1,\, \omega^q\neq 1} i_\omega(c) \geq \sum_{\omega^q=1} i_\omega(c) = i(c^q),$$

and

$$\nu(c^p) = \sum_{\omega^p=1} \nu_\omega(c) = \sum_{\omega^q=1} \nu_\omega(c) + \sum_{\omega^p=1,\, \omega^q\neq 1} \nu_\omega(c) \geq \sum_{\omega^q=1} \nu_\omega(c) = \nu(c^q).$$

Here that both $i_\omega(c)$ and $\nu_\omega(c)$ are non-negative integers for any $\omega \in \mathbf{U}$ follow from Proposition 1.3 of [Bot1]. The proof is complete. ∎

The following theorem gives some precise index properties of iterates of closed geodesics, which is a crucial step in the proofs of Theorems 1.1 and 1.2, and generalizes Theorem 3.7 in [LoD1] for rational closed geodesics to the non-rational closed geodesics.

Theorem 4.3. (**Index quasi-periodicity of closed geodesics**) *Let c be an orientable closed geodesic with $\hat{i}(c) > 0$ on a Finsler manifold of dimension d. Denote the basic normal form decomposition of the linearized Poincaré map P_c of c by (3.5). Let $n = n(c)$ be the analytical period of c.*

Then when $k \geq 1$, there exist an integer A with $[(k+1)/2] \leq A \leq k$ and a subset P of integers $\{1, \ldots, k\}$ with A integers such that for any given integer $m_0 \in \mathbf{N}$ and any small $\epsilon > 0$ there exists a sufficiently large even integer $T \in n\mathbf{N}$ satisfying

$$\left\{\frac{T\theta_j}{2\pi}\right\} > 1 - \epsilon, \qquad \text{for } j \in P, \tag{4.5}$$

$$\left\{\frac{T\theta_j}{2\pi}\right\} < \epsilon, \qquad \text{for } j \in \{1, \ldots, k\} \setminus P. \tag{4.6}$$

Note that both of (4.5) and (4.6) should be omitted when $k = 0$, and the following conclusions (A)-(D) still hold by Theorem 3.7 of [LoD1].

Let

$$p(c) \equiv p_- + p_0 + q_0 + q_+ + 2r_* - 2k_* + r + 2A - 2k \geq 0. \tag{4.7}$$

Then the following conclusions hold always:

(A) (Quasi-periodicity) For any $1 \le m \le m_0$, there hold

$$i(c^{m+T}) = i(c^m) + i(c^T) + p(c), \tag{4.8}$$

$$\nu(c^{m+T}) = \nu(c^m). \tag{4.9}$$

(B) (Relative parity) There holds

$$i(c^T) = p(c) \pmod 2. \tag{4.10}$$

(C) (Nullity-periodicity) There holds

$$\nu(c^n) = \nu(c^T) \le p(c) + d - 1 - 2A. \tag{4.11}$$

(D) (Period-mean index) If $\hat{i}(c) > 0$ is a rational number, there holds

$$T\hat{i}(c) = i(c^T) + p(c). \tag{4.12}$$

If $\hat{i}(c) > 0$ is irrational, then for any small $\tau > 0$ we can further require the above chosen $T \in n\mathbf{N}$ to be even larger to satisfy

$$|T\hat{i}(c) - (i(c^T) + p(c))| < \tau. \tag{4.13}$$

Proof. Note that by the definitions of $E(\cdot)$ and $\varphi(\cdot)$ there hold

$$E(z+b) = z + E(b), \qquad \varphi(z+b) = \varphi(b) \quad \text{for } \ k \in \mathbf{Z}, \quad b \notin \mathbf{Z}, \tag{4.14}$$

$$E(b) + E(-b) = 1, \qquad \text{for } \ b \in (0,1), \tag{4.15}$$

$$E(a) + E(-a) - \varphi(a) = 0, \qquad \varphi(a) = \varphi(-a) \quad \forall\, a \in \mathbf{R}. \tag{4.16}$$

Let $n = n(c)$ be the analytical period of c. For the integer m_0 given in the assumption of the theorem, we specially set

$$\epsilon = \min\left\{\{\frac{m\theta_j}{2\pi}\},\, 1 - \{\frac{m\theta_j}{2\pi}\}\,|\,1 \le m \le m_0;\, 1 \le j \le k\right\}. \tag{4.17}$$

Note that, when $k \ge 1$, we fix an even integer $T \in n\mathbf{N}$ obtained from Proposition 3.7 satisfying (4.5) and (4.6) for this $\epsilon > 0$.

We use short hand notations as in (3.5) and carry out the proof in several steps.

Step 1. *Proof of the quasi-periodicity (A).*

By (3.7) of Theorem 3.3 for any $m \in \mathbf{N}$ we obtain

$$\begin{aligned} i(c^{m+T}) &= (m+T)(i(c) + p_- + p_0 - r) + 2\sum_{j=1}^{r} E(\frac{(m+T)\theta_j}{2\pi}) - r \\ &\quad - p_- - p_0 - \frac{1+(-1)^m}{2}(q_0 + q_+) + 2\sum_{j=k_*+1}^{r_*} \varphi(\frac{m\alpha_j}{2\pi}) - 2(r_* - k_*) \\ &= i(c^m) + i(c^T) + (r + p_- + p_0 + q_0 + q_+ + 2r_* - 2k_*) \\ &\quad + 2\sum_{j=1}^{r} E(\frac{(m+T)\theta_j}{2\pi}) - 2\sum_{j=1}^{r} E(\frac{m\theta_j}{2\pi}) - 2\sum_{j=1}^{r} E(\frac{T\theta_j}{2\pi}). \end{aligned} \tag{4.18}$$

where we have used the evenness of T and the fact $T \in n\mathbf{N}$. Note that

$$
\begin{aligned}
\frac{1}{2}\Theta(m,T) &\equiv \sum_{j=1}^{r} E(\frac{(m+T)\theta_j}{2\pi}) - \sum_{j=1}^{r} E(\frac{m\theta_j}{2\pi}) - \sum_{j=1}^{r} E(\frac{T\theta_j}{2\pi}) \\
&= \sum_{j=1}^{k} E(\frac{(m+T)\theta_j}{2\pi}) - \sum_{j=1}^{k} E(\frac{m\theta_j}{2\pi}) - \sum_{j=}^{k} E(\frac{T\theta_j}{2\pi}) \\
&= \sum_{j=1}^{k} E\left(\{\frac{m\theta_j}{2\pi}\} + \{\frac{T\theta_j}{2\pi}\}\right) - \sum_{j=1}^{k} E\left(\{\frac{m\theta_j}{2\pi}\}\right) - \sum_{j=1}^{k} E\left(\{\frac{T\theta_j}{2\pi}\}\right) \\
&= \sum_{j=1}^{k} E\left(\{\frac{m\theta_j}{2\pi}\} + \{\frac{T\theta_j}{2\pi}\}\right) - 2k \\
&= \sum_{j=1}^{A} E\left(\{\frac{m\theta_j}{2\pi}\} + \{\frac{T\theta_j}{2\pi}\}\right) + \sum_{j=A+1}^{k} E\left(\{\frac{m\theta_j}{2\pi}\} + \{\frac{T\theta_j}{2\pi}\}\right) - 2k. \qquad (4.19)
\end{aligned}
$$

So it follows from (4.5)-(4.6) and (4.19) that

$$\Theta(m,T) = 2(A-k), \qquad \forall\, 1 \le m \le m_0. \qquad (4.20)$$

Together with (4.18), it yields

$$i(c^{m+T}) = i(c^m) + i(c^T) + r + p_- + p_0 + q_0 + q_+ + 2(r_* - k_*) + 2(A-k), \quad \forall\, 1 \le m \le m_0. \quad (4.21)$$

Thus (4.8) holds. And (4.9) follows from the definition of $T \in n\mathbf{N}$.

Step 2. *Proof of the relative parity (B).*

By Theorem 3.3 and the definition (4.7) of $p(c)$ we have

$$
\begin{aligned}
i(c^T) - p(c) &= T(i(c) + p_- + p_0 - r) + 2\sum_{j=1}^{r} E(\frac{T\theta_j}{2\pi}) \\
&\quad - r - p_- - p_0 - \frac{1+(-1)^T}{2}(q_0 + q_+) - 2(r_* - k_*) \\
&\quad - (p_- + p_0 + q_+ + q_0 + 2r_* - 2k_* + r + 2A - 2k) \\
&= T(i(c) + p_- + p_0 - r) + 2\sum_{j=1}^{r} E(\frac{T\theta_j}{2\pi}) - 2r - 2p_- - 2p_0 \\
&\quad - \frac{3+(-1)^T}{2}(q_0 + q_+) - 4(r_* - k_*) - 2(A-k).
\end{aligned}
$$

Because T is even, it yields the relative parity (B).

Step 3. *Proof of the nullity-periodicity (C).*

Because $\nu(c) = p_- + 2p_0 + p_+$, by Theorem 3.3 we have

$$
\begin{aligned}
\nu(c^T) - p(c) &= \nu(c^n) - p(c) \\
&= p_- + 2p_0 + p_+ + (q_- + 2q_0 + q_+) + 2(r - k + r_* - k_* + r_0 - k_0)
\end{aligned}
$$

$$
\begin{aligned}
& -(p_0+p_-+q_0+q_++2r_*-2k_*+r+2A-2k) \\
= \; & p_0+p_++q_0+q_-+r+2r_0-2k_0-2A \\
\le \; & d-1-2A.
\end{aligned}
$$

This yields (C).

Step 4. *Proof of the period-mean index (D).*

When $k=0$, (D) is proved in Theorem 3.7 of [LoD1]. Now we consider the case $k\ge 1$.

When $\hat{i}(c)=i(c)+p_-+p_0-r+\sum_{j=1}^{r}\theta_j/\pi$ is a rational number, we must have $k\ge 2$ and then $A\ge 1$ by Proposition 3.7. Let $\sum_{j=1}^{k}\theta_j/2\pi=q/p$ for some integers $0<p,q\in\mathbf{N}$ with $(p,q)=1$. Further choose $0<\epsilon<\frac{1}{k}$ and an even $T\in np\mathbf{N}$ satisfying (4.5) and (4.6). Note that $\sum_{j=1}^{k}\{\frac{T\theta_j}{2\pi}\}$ is an integer because T is an integer multiple of p.

If $A=k\ge 2$, by (4.5) it yields a contradiction

$$
\sum_{j=1}^{k}\{\frac{T\theta_j}{2\pi}\}=\sum_{j=1}^{A}\{\frac{T\theta_j}{2\pi}\}\in(A(1-\epsilon),A)\cap\mathbf{Z}=\emptyset. \tag{4.22}
$$

If $[\frac{k+1}{2}]\le A\le k-1$, by (4.5) and (4.6) we obtain

$$
\sum_{j=1}^{k}\{\frac{T\theta_j}{2\pi}\}\in(A(1-\epsilon),A+(k-A)\epsilon)\cap\mathbf{Z}=\{A\}. \tag{4.23}
$$

Together with (3.7) of Theorem 3.3 and the definition (4.7) of $p(c)$, it yields

$$
\begin{aligned}
T\hat{i}(c) &= T\left(i(c)+p_-+p_0-r+\sum_{j=1}^{r}\frac{\theta_j}{\pi}\right) \\
&= i(c^T)+(r+p_-+p_0+2r_*-2k_*) \\
&\quad +\frac{1+(-1)^T}{2}(q_0+q_+)+2\left(\sum_{j=1}^{k}\frac{T\theta_j}{2\pi}-\sum_{j=1}^{k}E(\frac{T\theta_j}{2\pi})\right) \\
&= i(c^T)+(r+p_-+p_0+2r_*-2k_*+q_0+q_+) \\
&\quad +2\left(\sum_{j=1}^{k}\{\frac{T\theta_j}{2\pi}\}-\sum_{j=1}^{k}E(\{\frac{T\theta_j}{2\pi}\})\right) \\
&= i(c^T)+(r+p_-+p_0+2r_*-2k_*+q_0+q_++2A-2k) \\
&= i(c^T)+p(c),
\end{aligned} \tag{4.24}
$$

where we have used the fact that $T\in np\mathbf{N}$ is even, and the rationality of $\hat{i}(c)$ is used only to get the second last equality.

When $\hat{i}(c)>0$ is irrational, we further require that $\epsilon>0$ satisfies $2k\epsilon<\tau$. Thus in this case (4.22) and (4.23) become

$$
\sum_{j=1}^{k}\{\frac{T\theta_j}{2\pi}\}\in(A(1-\epsilon),A+k\epsilon)\subset(A-\tau/2,A+\tau/2).
$$

Then from the third equality of (4.24) we obtain

$$|T\hat{i}(c) - (i(c^T) + p(c))| \le 2|\sum_{j=1}^{k}\{\frac{T\theta_j}{2\pi}\} - A| < \tau.$$

i.e., (D) holds.

This completes the proof of Theorem 4.3. ∎

Lemma 4.4. *For any orientable closed geodesic c with $\hat{i}(c) > 0$ on a Finsler manifold (M, F) of dimension d, denote the basic normal form decomposition of the linearized Poincaré map P_c of c by (3.5). Then for any even $T \in n\mathbf{N}$ and $m_0 = m_0(c)$ given by Definition 4.1, there holds*

$$i(c^{T+m_0+m}) - i(c^T) \ge p(c) + d, \qquad \forall\, m \ge 1. \tag{4.25}$$

Proof. By (3.7) of Theorem 3.3 and Definition 4.1, we obtain

$$\begin{aligned}
i(c^{T+m_0+m}) &= i(c^T) + (m+m_0)(i(c) + p_- + p_0 - r) + \frac{1-(-1)^{m_0+m}}{2}(q_0+q_+) \\
&\quad +2\sum_{j=1}^{r}\left(E(\frac{(m_0+m+T)\theta_j}{2\pi}) - E(\frac{T\theta_j}{2\pi})\right) + 2\sum_{j=k_*+1}^{r_*}\varphi(\frac{(m_0+m)\alpha_j}{2\pi}) \\
&= i(c^T) + i(c^{m_0+m}) + r + p_- + p_0 + 2(r_* - k_*) + q_0 + q_+ + 2A - 2k - (2A - 2k) \\
&\quad +2\sum_{j=1}^{r}\left(E(\frac{(m_0+m+T)\theta_j}{2\pi}) - E(\frac{(m_0+m)\theta_j}{2\pi}) - E(\frac{T\theta_j}{2\pi})\right) \\
&= i(c^T) + i(c^{m_0+m}) + p(c) + 2k - 2A \\
&\quad +2\sum_{j=1}^{k}\left(E(\{\frac{m_0\theta_j}{2\pi}\} + \{\frac{m\theta_j}{2\pi}\} + \{\frac{T\theta_j}{2\pi}\}) - E(\{\frac{m_0\theta_j}{2\pi}\} + \{\frac{m\theta_j}{2\pi}\}) - E(\{\frac{T\theta_j}{2\pi}\})\right) \\
&\ge i(c^T) + i(c^{m_0+m}) + p(c) - 2\sum_{j=1}^{k}E(\{\frac{m_0\theta_j}{2\pi}\} + \{\frac{m\theta_j}{2\pi}\}) \\
&\ge i(c^T) + i(c^{m_0+m}) + p(c) - 4k \\
&\ge i(c^T) + p(c) + d, \qquad \forall\, m \ge 1.
\end{aligned} \tag{4.26}$$

This completes the proof of Lemma 4.4. ∎

Next result generalizes Proposition 3.11 of [LoD1] for rational closed geodesics to irrational ones.

Theorem 4.5. *For every orientable closed geodesic c with $\hat{i}(c) > 0$ on a Finsler manifold (M, F) of dimension $d \ge 2$, denote the basic normal form decomposition of the linearized Poincaré map P_c of c by (3.5). Then there exist an integer A with $[(k+1)/2] \le A \le k$ and a subset P of integers $\{1, \ldots, k\}$ with A integers such that for any small $\epsilon > 0$ there exists a sufficiently large even integer $T \in n\mathbf{N}$ such that (4.5) and (4.6) and the following estimate hold*

$$i(c^m) + \nu(c^m) \le i(c^T) + p(c) + d - 3, \qquad \forall\, 1 \le m \le T-1. \tag{4.27}$$

Proof. When $k=0$, i.e., the closed geodesic c is rational, this result was proved in Proposition 3.11 of [LoD1], whose proof there in fact did not use the fact $^{\#}\mathrm{CG}(M,F)=1$. Therefore here we only consider the case of $k\geq 1$.

On the one hand, by Theorem 3.3, for any $1\leq m\leq T-1$, we have

$$\begin{aligned}
i(c^m) \quad &+ \quad i(c^{T-m})+\nu(c^m) \\
&= \quad i(c^T)+2\sum_{j=1}^{r}\left(E(\frac{m\theta_j}{2\pi})+E(\frac{(T-m)\theta_j}{2\pi})-E(\frac{T\theta_j}{2\pi})\right)-(r+p_-+p_0) \\
&\quad -(-1)^m(q_0+q_+)+2\sum_{j=k_*+1}^{r_*}\varphi(-\frac{m\alpha_j}{2\pi})+2\sum_{j=k_*+1}^{r_*}\varphi(\frac{m\alpha_j}{2\pi})-2(r_*-k_*) \\
&\quad +p_-+2p_0+p_++\frac{1+(-1)^m}{2}(q_-+2q_0+q_+)+2(r-k+r_*-k_*+r_0-k_0) \\
&\quad -2[\sum_{j=k+1}^{r}\varphi\left(\frac{m\theta_j}{2\pi}\right)+\sum_{j=k_*+1}^{r_*}\varphi\left(\frac{m\alpha_j}{2\pi}\right)+\sum_{j=k_0+1}^{r_0}\varphi\left(\frac{m\beta_j}{2\pi}\right)] \\
&= \quad i(c^T)+2\sum_{j=1}^{r}\left(E(\frac{m\theta_j}{2\pi})+E(\frac{(T-m)\theta_j}{2\pi})-E(\frac{T\theta_j}{2\pi})\right)-r+p_0+p_+ \\
&\quad -(-1)^m(q_0+q_+)+\frac{1+(-1)^m}{2}(q_-+2q_0+q_+)+2(r-k+r_0-k_0) \\
&\quad +2\sum_{j=k_*+1}^{r_*}\varphi(-\frac{m\alpha_j}{2\pi})-2[\sum_{j=k+1}^{r}\varphi\left(\frac{m\theta_j}{2\pi}\right)+\sum_{j=k_0+1}^{r_0}\varphi\left(\frac{m\beta_j}{2\pi}\right)], \qquad (4.28)
\end{aligned}$$

where we have used the fact $T\in n\mathbf{N}$ is even and the fact $\nu(c)=p_-+p_++2p_0$ by the definitions of p_*s in Theorem 3.2.

Note that by (4.16) we get

$$\begin{aligned}
&2\sum_{j=1}^{r}\left(E(\frac{m\theta_j}{2\pi})+E(\frac{(T-m)\theta_j}{2\pi})-E(\frac{T\theta_j}{2\pi})\right)-2\sum_{j=k+1}^{r}\varphi\left(\frac{m\theta_j}{2\pi}\right) \\
&\quad =2\sum_{j=1}^{k}\left(E(\frac{m\theta_j}{2\pi})+E(\frac{(T-m)\theta_j}{2\pi})-E(\frac{T\theta_j}{2\pi})\right) \\
&\qquad +2\sum_{j=k+1}^{r}\left(E(\frac{m\theta_j}{2\pi})+E(-\frac{m\theta_j}{2\pi})\right)-2\sum_{j=k+1}^{r}\varphi\left(\frac{m\theta_j}{2\pi}\right) \\
&\quad =2\sum_{j=1}^{k}\left(E(\{\frac{m\theta_j}{2\pi}\})+E(\{\frac{T\theta_j}{2\pi}\}-\{\frac{m\theta_j}{2\pi}\})-E(\{\frac{T\theta_j}{2\pi}\})\right) \\
&\quad =2\sum_{j=1}^{k}\left(E(\{\frac{T\theta_j}{2\pi}\}-\{\frac{m\theta_j}{2\pi}\})\right) \\
&\quad \leq 2k. \qquad (4.29)
\end{aligned}$$

Together with (4.28), it yields

$$i(c^m) \quad + \quad i(c^{T-m})+\nu(c^m)$$

$$
\begin{aligned}
&\leq i(c^T)+2k-r+p_0+p_++2(r-k+r_0-k_0)-(-1)^m(q_0+q_+)\\
&\quad+\frac{1+(-1)^m}{2}(q_-+2q_0+q_+)+2\sum_{j=k_*+1}^{r_*}\varphi(\frac{m\alpha_j}{2\pi})-2\sum_{j=k_0+1}^{r_0}\varphi\left(\frac{m\beta_j}{2\pi}\right)\\
&=i(c^T)+r+p_0+p_++q_0+2(r_0-k_0)+\frac{1-(-1)^m}{2}q_+\\
&\qquad+\frac{1+(-1)^m}{2}q_-+2\sum_{j=k_*+1}^{r_*}\varphi(\frac{m\alpha_j}{2\pi})-2\sum_{j=k_0+1}^{r_0}\varphi\left(\frac{m\beta_j}{2\pi}\right)\\
&\leq i(c^T)+(p_-+p_0+q_0+q_++2r_*-2k_*+r+2A-2k)+p_++2(r_0-k_0)\\
&\qquad-p_--2(A-k)-\frac{1+(-1)^m}{2}q_++\frac{1+(-1)^m}{2}q_--2\sum_{j=k_0+1}^{r_0}\varphi\left(\frac{m\beta_j}{2\pi}\right)\\
&\leq i(c^T)+p(c)+p_++q_-+2r_0-2(A-k)\\
&\leq i(c^T)+p(c)+p_++q_-+2r_0+k, \qquad \forall\, 1\leq m\leq T-1. \qquad (4.30)
\end{aligned}
$$

In other words, we obtain

$$
i(c^m)+\nu(c^m)\leq i(c^T)+p(c)-i(c^{T-m})+p_++q_-+2r_0+k, \qquad 1\leq m\leq T-1. \tag{4.31}
$$

On the other hand, it follows from Theorem 3.8 that

$$
\begin{aligned}
i(c^m) &\leq i(c^T)-i(c)-p_0-p_-+r+q_0+q_++2(r_*-k_*)+2(A-k)\\
&= i(c^T)+p(c)-i(c)-2(p_0+p_-), \qquad \forall\, 1\leq m\leq T-1. \qquad (4.32)
\end{aligned}
$$

Note that $p_++q_-+2r_0+k\leq d-1$ holds always in (4.31) by (3.6) with d replaced by $d-1$. If $p_++q_-+2r_0+k\leq d-3$, then (4.31) yields (4.27). Therefore to continue our proof, it suffices to consider the following two distinct cases.

Case 1. $p_++q_-+2r_0+k=d-1$.

In this case, by (3.8) and (3.10) for all $m\geq 1$ we have

$$
\nu(c^m)\leq\nu(c^n)=p_++q_-+2r_0=d-1-k. \tag{4.33}
$$

Thus together with (4.32) it yields

$$
i(c^m)+\nu(c^m)\leq i(c^T)+p(c)-i(c)+d-1-k. \tag{4.34}
$$

So in order to prove (4.27), it suffices to consider the case of $i(c)=0$ and $k=1$. By the fact $i(c)=0$ and Proposition 3.4, we must have $q_-\in 2\mathbf{N}-1$ and thus $n\in 2\mathbf{N}$ by the definition of $n=n(c)$.

Therefore we have

$$
P_c\approx N_1(1,-1)^{\diamond p_+}\diamond N_1(-1,1)^{\diamond q_-}\diamond(\diamond_{j=1}^{r_0}N_2(e^{\beta_j\sqrt{-1}},B_j))\diamond R(\theta_1), \tag{4.35}
$$

where $\theta_1/\pi \in (0,2) \setminus \mathbf{Q}$. Thus by Theorem 3.3, we have

$$\begin{aligned} i(c^m) &= -m + 2E(\frac{m\theta_1}{2\pi}) - 1, \qquad \forall\, m \in \mathbf{N}, & (4.36)\\ \nu(c^n) &= p_+ + q_- + 2r_0 = d - 1 - k = d - 2. & (4.37) \end{aligned}$$

When $m \in (\mathbf{N} \setminus n\mathbf{N})$, we have $\nu(c^m) < \nu(c^n) = \nu(c^T)$. Thus by (4.32) and (4.37) we get

$$i(c^m) + \nu(c^m) \le i(c^T) + p(c) + \nu(c^n) - 1 = i(c^T) + p(c) + d - 3,$$

i.e., (4.27) holds.

Then for $1 \le mn < T$ and the T chosen above, by $\hat{i}(c) > 0$ we get

$$\begin{aligned} i(c^T) - i(c^{mn}) &= mn - T + 2\left(E(\frac{T\theta_1}{2\pi}) - E(\frac{mn\theta_1}{2\pi})\right)\\ &= mn - T + (T - mn)\frac{\theta_1}{\pi} + 2\left(\{\frac{mn\theta_1}{2\pi}\} - \{\frac{T\theta_1}{2\pi}\}\right)\\ &= (T - mn)\hat{i}(c) + 2\left(\{\frac{mn\theta_1}{2\pi}\} - \{\frac{T\theta_1}{2\pi}\}\right)\\ &> 2\left(\{\frac{mn\theta_1}{2\pi}\} - \{\frac{T\theta_1}{2\pi}\}\right)\\ &> -2. \end{aligned} \tag{4.38}$$

Since both n and T are even, it follows from (4.36) that $i(c^T) - i(c^{mn})$ is even. Thus by the irrationality of $\frac{\theta_1}{\pi}$ and (4.38) we obtain

$$i(c^T) \ge i(c^{mn}), \qquad \forall\, 1 \le mn < T. \tag{4.39}$$

Then by the fact $p(c) = 1$ and (4.37)-(4.39) we have

$$i(c^{mn}) + \nu(c^{mn}) \le i(c^T) + \nu(c^n) \le i(c^T) + p(c) - 1 + d - 2 = i(c^T) + p(c) + d - 3. \tag{4.40}$$

That is, (4.27) holds.

Case 2. $p_+ + q_- + 2r_0 + k = d - 2$

In this case, $p_- + p_0 + q_0 + q_+ + r - k + 2r_* + h_- + h_+ = 1$ by (3.6) with d replaced by $d-1$, which implies $r_* = 0$. By Theorem 3.3 we have

$$\begin{aligned} \nu(c^m) &\le \nu(c^n)\\ &= p_+ + q_- + 2r_0 + (p_- + q_+ + 2p_0 + 2q_0 + 2(r - k))\\ &= d - 2 - k + (p_- + q_+ + 2p_0 + 2q_0 + 2(r - k)), \quad \forall\, m \ge 1. \end{aligned} \tag{4.41}$$

If $k \ge 3$, by (4.41) it yields $\nu(c^m) \le d - 3$, $\forall\, m \ge 1$. Thus together with (4.32), it yields (4.27). If $k = 2$, by (4.32) and (4.41) it suffices to consider the following case

$$i(c) = p_0 = p_- = q_+ = h_+ = h_- = r_* = 0, \quad q_0 + (r - k) = 1, \quad k = 2, \tag{4.42}$$

because otherwise (4.32) would imply (4.27) already.

Similarly, if $k=1$, by (4.31), (4.32) and (4.41) it suffices to consider the following case

$$i(c^{T-m}) = p_0 = p_- = h_+ = h_- = r_* = 0, \quad q_+ + q_0 + (r-k) = 1, \quad k = 1. \tag{4.43}$$

because otherwise (4.31) and (4.32) would imply (4.27) already.

Now we consider (4.42) and (4.43) respectively.

Case 2.1. *(4.42) happens.*

Viewing $-I$ as $R(\pi)$ if $q_0 = 1$, it suffices to consider the case $r-k=1$. Thus in addition to (4.42) we have

$$r = 3, \qquad q_0 = 0. \tag{4.44}$$

Therefore we have

$$P_c \approx N_1(1,-1)^{\diamond p_+} \diamond N_1(-1,1)^{\diamond q_-} \diamond (\diamond_{j=1}^{r_0} N_2(e^{\beta_j\sqrt{-1}}, B_j)) \diamond R(\theta_1) \diamond R(\theta_2) \diamond R(\theta_3), \tag{4.45}$$

where θ_1/π and $\theta_2/\pi \in (0,2)\setminus \mathbf{Q}$ and $\theta_3/\pi \in (0,2)\cap\mathbf{Q}$. Thus by Theorem 3.3, we have

$$i(c^m) = -3m + 2\sum_{j=1}^{3} E(\frac{m\theta_j}{2\pi}) - 3, \qquad \forall\, m \in \mathbf{N}, \tag{4.46}$$

$$\nu(c^n) = p_+ + q_- + 2r_0 + 2 = d - 2. \tag{4.47}$$

By the fact $i(c) = 0$, (4.45) and Proposition 3.4, there holds $q_- \in 2\mathbf{N}-1$. By the definition of n, it further yields

$$n \in 2\mathbf{N}. \tag{4.48}$$

When $m \in (\mathbf{N}\setminus n\mathbf{N})$, we have $\nu(c^m) < \nu(c^n) = \nu(c^T)$. Thus by (4.32) and (4.47) we get

$$i(c^m) + \nu(c^m) \le i(c^T) + p(c) + \nu(c^n) - 1 = i(c^T) + p(c) + d - 3,$$

i.e., (4.27) holds.

When $mn \in \mathbf{N}$ and $1 \le mn < T$, then by (4.46) and (4.48) we have $i(c^{T-mn}) \in 2\mathbf{N}-1$. Therefore by (4.31), for any $1 \le mn < T$, we get

$$\begin{aligned} i(c^{mn}) + \nu(c^{mn}) &\le i(c^T) + p(c) - i(c^{T-mn}) + d - 2 \\ &\le i(c^T) + p(c) + d - 3. \end{aligned} \tag{4.49}$$

That is, (4.27) holds.

Case 2.2. *(4.43) happens.*

Viewing $-I$ (or $N_1(-1,-1)$) as $R(\pi)$ if $q_0 = 1$ (or $q_+ = 1$), although their nullity may be different by 1 (cf. (4.53) below), it suffices to consider the case $r-k=1$. Thus in addition to (4.43) we have

$$p(c) = 2, \quad r = 2, \quad q_+ = q_0 = 0. \tag{4.50}$$

Therefore we have

$$P_c \approx N_1(1,-1)^{\diamond p_+}\diamond N_1(-1,1)^{\diamond q_-}\diamond(\diamond_{j=1}^{r_0} N_2(e^{\beta_j\sqrt{-1}},B_j))\diamond R(\theta_1)\diamond R(\theta_2), \tag{4.51}$$

where $\theta_1/\pi \in (0,2)\setminus \mathbf{Q}$ and $\theta_2/\pi \in (0,2)\cap \mathbf{Q}$. Thus by Theorem 3.3, we have

$$\begin{aligned} i(c^m) &= (i(c)-2)m + 2\sum_{j=1}^{2} E(\frac{m\theta_j}{2\pi}) - 2, \qquad \forall\, m\in\mathbf{N}, &(4.52)\\ \nu(c^n) &= p_+ + q_- + 2r_0 + 2(r-k) = d-1. &(4.53)\end{aligned}$$

If $q_- \in 2\mathbf{N}-1$, by (4.51) and Proposition 3.4, there holds

$$n\in 2\mathbf{N} \qquad \text{and} \qquad i(c^{T-m}) \geq i(c) \in 2\mathbf{N}-1, \quad \forall\, 1\leq m\leq T-1. \tag{4.54}$$

Therefore, by (4.31), (4.50) and (4.53)-(4.54) we get

$$i(c^m)+\nu(c^m) \leq i(c^T)+p(c)-i(c^{T-m})+d-2 \leq i(c^T)+p(c)+d-3, \tag{4.55}$$

That is, (4.27) holds.

If $q_- \in 2\mathbf{N}$, by (4.51) and Proposition 3.4, it yields $i(c)\in 2\mathbf{N}_0$. So it follows from (4.52) that $i(c^m)\in 2\mathbf{N}_0$ for all $m\geq 1$. Let $\frac{q}{p}=\frac{\theta_2}{2\pi}$ with integers p and q satisfying $(p,q)=1$.

When $m\in(\mathbf{N}\setminus p\mathbf{N})$, we have $\nu(c^m)\leq \nu(c^n)-2=d-3$ by (4.53). Thus by (4.32) it yields

$$i(c^m)+\nu(c^m)\leq i(c^T)+p(c)+d-3. \tag{4.56}$$

When $m\in p\mathbf{N}$, then, similarly to (4.38), we can obtain $i(c^T)\geq i(c^m)$. Therefore, by (4.50) and (4.53), we get

$$i(c^m)+\nu(c^m)\leq i(c^T)+\nu(c^n)\leq i(c^T)+d-1+p(c)-2 = i(c)+p(c)+d-3, \tag{4.57}$$

That is, (4.27) holds.

The proof is complete. ∎

5 Homological quasi-periodicity

In this section, we study properties of homologies of energy level sets determined by closed geodesics and establish certain periodicity of homological modules of energy level set pairs when there exists only one prime closed geodesic.

For any $m\in\mathbf{N}$, denote the energy level $E(c^m)$ of c^m by

$$\kappa_m = E(c^m). \tag{5.1}$$

It is well known that $\kappa_m = E(c^m) = m^2E(c)$ is strictly increasing to $+\infty$. Set $\kappa_0 = 0$. The next lemma follows from Theorem 3 of [GrM1], the Theorem on p.367 of [GrM2], Lemma 3.1 to Theorem 3.7 of [Lon4], and Theorem I.4.2 of [Cha1].

Lemma 5.1. (Lemma 4.2 of [LoD1]) *Let $M = (M, F)$ be a Finsler manifold with* $\dim M < +\infty$. *Let c be a closed geodesic on M each of whose iteration $S^1 \cdot c^m$ is an isolated critical orbit of E in the loop space ΛM. Suppose that there are integers $m \in \mathbf{N}$ and $p \in 2\mathbf{N}_0$ such that*

$$i(c^m) = i(c) + p, \qquad \nu(c^m) = \nu(c). \tag{5.2}$$

Then the iteration map ψ^m induces an isomorphism

$$\psi^m_* : \overline{C}_*(E, c) \to \overline{C}_{*+p}(E, c^m). \tag{5.3}$$

One of the key results in [LoD1] is the homological isomorphism Theorem 4.3 there for rational closed geodesics. Below we redescribe this theorem and give more details on two points for the proof given in [LoD1].

Theorem 5.2. (Theorem 4.3 of [LoD1]) *Let $M = (M, F)$ be a Finsler manifold possessing only one prime closed geodesic c which is rational and orientable. Let $n = n(c)$ be the analytical period of c. Recall that by Theorem 3.7 of [LoD1] there hold*

$$i(c^{m+n}) = i(c^m) + \overline{p}, \quad \nu(c^{m+n}) = \nu(c^m), \qquad \forall\, m \in \mathbf{N}, \tag{5.4}$$

where $\overline{p} = i(c^n) + p(c)$ is even. Then for any non-negative integers $b > a$ and any integer $h \in \mathbf{Z}$, the iteration maps $\{\psi^m\}$ and inclusion maps of corresponding level sets induce a map f on singular chains which yields an isomorphism

$$f_* : H_h(\overline{\Lambda}^{\kappa_b}, \overline{\Lambda}^{\kappa_a}) \to H_{h+\overline{p}}(\overline{\Lambda}^{\kappa_{n+b}}, \overline{\Lambda}^{\kappa_{n+a}}). \tag{5.5}$$

Proof. Proof of this theorem was given in [LoD1] based on the above Lemma 5.1. Note that an important condition in Lemma 5.1 is that the constant p in (5.2) should be even. In the applications of Lemma 5.1 (i.e., Lemma 4.2 of [LoD1]) in the proof of Theorem 4.3 in [LoD1], there are two points in its Step 1 on which we did not give details on how to get this evenness condition. Below we provide details of the proofs for these two points.

Because there is only one prime closed geodesic c on M, and $\kappa_m = E(c^m) = m^2E(c) = m^2\kappa_1 > 0$ is strictly increasing to $+\infty$, the critical module of E at $S^1 \cdot c^m$ can be defined by

$$\overline{C}_j(E, c^m) = H_j(\overline{\Lambda}^{\kappa_m}, \overline{\Lambda}^{\kappa_m\#}) = H_j(\overline{\Lambda}^{\kappa_m}, \overline{\Lambda}^{\kappa_{m-1}}), \tag{5.6}$$

where and below we denote by

$$\overline{\Lambda}^{\kappa_m\#} \equiv \overline{\Lambda^{\kappa_m} \setminus (S^1 \cdot c^m)}. \tag{5.7}$$

Given a level set pair $(\overline{\Lambda}^{\kappa_p}, \overline{\Lambda}^{\kappa_p\#})$ with $p \in \mathbf{N}$, for any $\gamma \in \Lambda^{\kappa_p}$ and $m \in \mathbf{N}$ we have

$$E(\psi^m(\gamma)) = m^2 E(\gamma) \leq m^2 \kappa_p = m^2 E(c^p) = E(c^{mp}) = \kappa_{mp}.$$

Therefore the iteration map ψ^m maps the level set $\overline{\Lambda}^{\kappa_p}$ into $\overline{\Lambda}^{\kappa_{mp}}$. We denote the image of the pair $(\overline{\Lambda}^{\kappa_p}, \overline{\Lambda}^{\kappa_p\#})$ under the iteration map ψ^m by

$$(\overline{\Lambda}^{\kappa_p}, \overline{\Lambda}^{\kappa_p\#})^m = (\psi^m(\overline{\Lambda}^{\kappa_p}), \psi^m(\overline{\Lambda}^{\kappa_p\#})) = (\overline{\psi^m(\Lambda^{\kappa_p})}, \overline{\psi^m(\Lambda^{\kappa_p} \setminus (S^1 \cdot c^p))}\,). \tag{5.8}$$

Note that we have (cf. (4.13)-(4.16) of [LoD1])

$$b = kn + q \qquad \text{for some } k \in \mathbf{N}_0 \ \text{ and } \ 0 \leq q \leq n-1, \tag{5.9}$$

$$i(c^b) = k\overline{p} + i(c^q), \quad i(c^{n+b}) = (k+1)\overline{p} + i(c^q), \tag{5.10}$$

$$\nu(c^{n+b}) = \nu(c^b) = \nu(c^q), \quad \text{when } \ q \neq 0. \tag{5.11}$$

Point 1. *The Proof of Case (i) with $\nu(c^b) = \nu(c)$ on Page 1787 of [LoD1]*

Below (4.17) in Page 1787 of [LoD1], we have defined $\hat{p} = i(c^q) - i(c)$.

Now if $\hat{p}$ is even, then the constant $k\overline{p} + \hat{p}$ is even, and then we can use Lemma 5.1 to get the isomorphism (4.23) in Page 1787 of [LoD1]. Thus the proof on Page 1787 for the Case (i) in [LoD1] goes through.

Now if $\hat{p}$ is odd, then both q and $n = n(c)$ must be even by (4.3) of Lemma 4.2 and the definition of $n(c)$. Therefore b is even by (5.9). By (4.4) of Lemma 4.2 and (5.11) we then obtain

$$\nu(c^{n+b}) = \nu(c^b) = \nu(c^q) \geq \nu(c^2) \geq \nu(c).$$

Thus equalities must hold here and we get

$$\nu(c^{n+b}) = \nu(c^b) = \nu(c^q) = \nu(c^2). \tag{5.12}$$

We define $\tilde{p} = i(c^q) - i(c^2)$. Then $\tilde{p}$ is even by Lemma 4.2. We have also

$$i(c^b) \quad = \quad k\overline{p} + i(c^q) = k\overline{p} + \tilde{p} + i(c^2), \tag{5.13}$$

$$i(c^{n+b}) \quad = \quad (k+1)\overline{p} + i(c^q) = (k+1)\overline{p} + \tilde{p} + i(c^2). \tag{5.14}$$

Thus we can replace (4.18)-(4.23) in [LoD1] by the following arguments, and obtain that the two iteration maps

$$\psi^{b/2}: \qquad (\overline{\Lambda}^{\kappa_2}, \overline{\Lambda}^{\kappa_2\#}) \to (\overline{\Lambda}^{\kappa_2}, \overline{\Lambda}^{\kappa_2\#})^{b/2} \subseteq (\overline{\Lambda}^{\kappa_b}, \overline{\Lambda}^{\kappa_b\#}), \tag{5.15}$$

$$\psi^{(n+b)/2}: \qquad (\overline{\Lambda}^{\kappa_2}, \overline{\Lambda}^{\kappa_2\#}) \to (\overline{\Lambda}^{\kappa_2}, \overline{\Lambda}^{\kappa_2\#})^{(n+b)/2} \subseteq (\overline{\Lambda}^{\kappa_{n+b}}, \overline{\Lambda}^{\kappa_{n+b}\#}), \tag{5.16}$$

induce two isomorphisms on homological modules:

$$\psi_*^{b/2}: \qquad H_{h-k\overline{p}-\tilde{p}}(\overline{\Lambda}^{\kappa_2}, \overline{\Lambda}^{\kappa_2\#}) = \overline{C}_{h-k\overline{p}-\tilde{p}}(E, c^2) \to \overline{C}_h(E, c^b) = H_h(\overline{\Lambda}^{\kappa_b}, \overline{\Lambda}^{\kappa_b\#}), \tag{5.17}$$

$$\begin{aligned} \psi_*^{(n+b)/2}: \qquad & H_{h-k\overline{p}-\tilde{p}}(\overline{\Lambda}^{\kappa_2}, \overline{\Lambda}^{\kappa_2\#}) = \overline{C}_{h-k\overline{p}-\tilde{p}}(E, c^2) \\ & \to \overline{C}_{h+\overline{p}}(E, c^{n+b}) = H_{h+\overline{p}}(\overline{\Lambda}^{\kappa_{n+b}}, \overline{\Lambda}^{\kappa_{n+b}\#}). \end{aligned} \tag{5.18}$$

Therefore the composed iteration map

$$f = \psi^{(n+b)/2} \circ \psi^{-b/2} : (\overline{\Lambda}^{\kappa_2}, \overline{\Lambda}^{\kappa_2\#})^{b/2} \to (\overline{\Lambda}^{\kappa_2}, \overline{\Lambda}^{\kappa_2\#})^{(n+b)/2} \tag{5.19}$$

is a homeomorphism and induces an isomorphism on homological modules:

$$f_* : \qquad H_h(\overline{\Lambda}^{\kappa_b}, \overline{\Lambda}^{\kappa_b\#}) = \overline{C}_h(E, c^b) \to \overline{C}_{h+\overline{p}}(E, c^{n+b}) = H_{h+\overline{p}}(\overline{\Lambda}^{\kappa_{n+b}}, \overline{\Lambda}^{\kappa_{n+b}\#}), \tag{5.20}$$

where we denote by $\psi^{-h} = (\psi^h)^{-1}$, the inverse map of ψ^h. Thus Theorem 5.2 holds in this case.

Point 2. *The Proof of Case (iii-2) with $\nu(c^b) > \nu(c)$, $q > 0$ in (5.9), and that there is some integer $t \in [1, q-1]$ such that $t|q$, $t|n$ and $\nu(c^t) = \nu(c^q)$ hold, in Page 1789 of [LoD1].*

As in Page 1789 of [LoD1], let $s \in [1, q-1]$ be the minimal integer possessing the property of the above integer t. Then $q = us$ and $n = vs$ hold for some $u, v \in \mathbf{N}$, and as in [LoD1] we obtain

$$b = kn + (q-s) + s = (kv+u)s, \tag{5.21}$$

$$n + b = (k+1)n + (q-s) + s = ((k+1)v+u)s, \tag{5.22}$$

$$\nu(c^s) = \nu(c^q) = \nu(c^b) = \nu(c^{n+b}). \tag{5.23}$$

Let $\hat{p} = i(c^q) - i(c^s)$.

Now if $\hat{p}$ is even, then the constant $k\overline{p} + \hat{p}$ is even, and then we can use Lemma 5.1 to get the isomorphism (4.46) in Page 1789 of [LoD1]. Thus the proof on Page 1789 for the Case (iii-2) in [LoD1] goes through.

Now if $\hat{p}$ is odd, then $n = n(c)$ must be even by (4.3) of Lemma 4.2 and the definition of $n(c)$. By the same reason, s and q must have different parity.

Now if s is even, then q must be odd. Thus b is odd by the evenness of n and (5.9). This contradicts to (5.21). Therefore s must be odd and q is even.

Because s is odd, and both $s|q$ and $2|q$ hold, we have $(2s)|q$. Similarly $s|n$ and $2|n$ imply $(2s)|n$. Then by (5.9) we obtain $(2s)|b$ and $(2s)|(n+b)$.

On the other hand, by (4.4) of Lemma 4.2 and the fact $(2s)|q$, we obtain

$$\nu(c^q) \geq \nu(c^{2s}) \geq \nu(c^s).$$

Together with (5.23) we then obtain

$$\nu(c^{n+b}) = \nu(c^b) = \nu(c^q) = \nu(c^{2s}) = \nu(c^s). \tag{5.24}$$

In this case we define $\tilde{p} = i(c^q) - i(c^{2s})$. Then $\tilde{p}$ is even by Lemma 4.2. We have also

$$i(c^b) \quad = \quad k\overline{p} + i(c^q) = k\overline{p} + \tilde{p} + i(c^{2s}), \tag{5.25}$$

$$i(c^{n+b}) \quad = \quad (k+1)\overline{p} + i(c^q) = (k+1)\overline{p} + \tilde{p} + i(c^{2s}). \tag{5.26}$$

Thus we can replace (4.41)-(4.46) in [LoD1] by the following arguments, and obtain from Lemma 5.1 that the two iteration maps

$$\psi^{b/(2s)}: \quad (\overline{\Lambda}^{\kappa_{2s}}, \overline{\Lambda}^{\kappa_{2s}\#}) \to (\overline{\Lambda}^{\kappa_{2s}}, \overline{\Lambda}^{\kappa_{2s}\#})^{b/(2s)} \subseteq (\overline{\Lambda}^{\kappa_b}, \overline{\Lambda}^{\kappa_b\#}), \tag{5.27}$$

$$\psi^{(n+b)/(2s)}: \quad (\overline{\Lambda}^{\kappa_{2s}}, \overline{\Lambda}^{\kappa_{2s}\#}) \to (\overline{\Lambda}^{\kappa_{2s}}, \overline{\Lambda}^{\kappa_{2s}\#})^{(n+b)/(2s)} \subseteq (\overline{\Lambda}^{\kappa_{n+b}}, \overline{\Lambda}^{\kappa_{n+b}\#}), \tag{5.28}$$

induce two isomorphisms on homological modules:

$$\psi_*^{b/(2s)}: \quad H_{h-k\overline{p}-\tilde{p}}(\overline{\Lambda}^{\kappa_{2s}}, \overline{\Lambda}^{\kappa_{2s}\#}) = \overline{C}_{h-k\overline{p}-\tilde{p}}(E, c^{2s}) \to \overline{C}_h(E, c^b) = H_h(\overline{\Lambda}^{\kappa_b}, \overline{\Lambda}^{\kappa_b\#}), \tag{5.29}$$

$$\begin{aligned} \psi_*^{(n+b)/(2s)}: \quad & H_{h-k\overline{p}-\tilde{p}}(\overline{\Lambda}^{\kappa_{2s}}, \overline{\Lambda}^{\kappa_{2s}\#}) = \overline{C}_{h-k\overline{p}-\tilde{p}}(E, c^{2s}) \\ & \to \overline{C}_{h+\overline{p}}(E, c^{n+b}) = H_{h+\overline{p}}(\overline{\Lambda}^{\kappa_{n+b}}, \overline{\Lambda}^{\kappa_{n+b}\#}). \end{aligned} \tag{5.30}$$

Therefore the composed iteration map

$$f = \psi^{(n+b)/(2s)} \circ \psi^{-b/(2s)}: (\overline{\Lambda}^{\kappa_{2s}}, \overline{\Lambda}^{\kappa_{2s}\#})^{b/(2s)} \to (\overline{\Lambda}^{\kappa_{2s}}, \overline{\Lambda}^{\kappa_{2s}\#})^{(n+b)/(2s)} \tag{5.31}$$

is a homeomorphism and induces an isomorphism on homological modules:

$$f_*: H_h(\overline{\Lambda}^{\kappa_b}, \overline{\Lambda}^{\kappa_b\#}) = \overline{C}_h(E, c^b) \to \overline{C}_{h+\overline{p}}(E, c^{n+b}) = H_{h+\overline{p}}(\overline{\Lambda}^{\kappa_{n+b}}, \overline{\Lambda}^{\kappa_{n+b}\#}). \tag{5.32}$$

Thus Theorem 5.2 holds in this case too.

Now the rest part of the proof of Theorem 4.3 of [LoD1] yields Theorem 5.2. ∎

The above homological isomorphism theorem is for rational closed geodesics. Our next result generalizes it to irrational closed geodesics, and will play a crucial role in the proofs of Theorems 1.1 and 1.2. Here the quasi-periodicity which we established in the above Theorem 4.3 is crucial in the proof.

Theorem 5.3. *Let (M, F) be a Finsler manifold possessing only one prime closed geodesic c which is orientable and $n = n(c)$ be the analytical period of c. Recall that by Theorem 4.3 there exists an even integer $T \in n\mathbf{N}$ such that for $m_0 = m_0(c)$ given by Definition 4.1 there hold*

$$i(c^{m+T}) = i(c^m) + \overline{p}, \quad \nu(c^{m+T}) = \nu(c^m), \qquad \forall\, 1 \le m \le m_0, \tag{5.33}$$

where $\overline{p} = i(c^T) + p(c)$. Then we can further require $T \in (m_0!n)\mathbf{N}$ such that for any non-negative integers a and b satisfying $0 < a < b \le m_0$, the iteration maps $\{\psi^m\}$ and inclusion maps of corresponding level sets induce a map f on singular chains which yields an isomorphism

$$f_*: H_h(\overline{\Lambda}^{\kappa_b}, \overline{\Lambda}^{\kappa_a}) \to H_{h+\overline{p}}(\overline{\Lambda}^{\kappa_{T+b}}, \overline{\Lambda}^{\kappa_{T+a}}), \qquad \forall\, h \in \mathbf{Z}. \tag{5.34}$$

Proof. Here we follow the main ideas from pages 1786-1792 of [LoD1].

Step 1. *The isomorphism in the case of $b - a = 1$.*

In this case κ_a and κ_b are the only two critical values in $[\kappa_a, \kappa_b]$ and so are κ_{T+a} and κ_{T+b} in $[\kappa_{T+a}, \kappa_{T+b}]$. Then, note that $0 \le a < b \le m_0$, by (5.33) it yields

$$i(c^{T+b}) = \overline{p} + i(c^b), \qquad \nu(c^{T+b}) = \nu(c^b). \tag{5.35}$$

Here we require the even integer T chosen by Theorems 4.3-4.5 to further satisfy $T \in (m_0!n)\mathbf{N}$. Thus it yields

$$b\,|(T+b), \qquad \forall\, 0 \le a < b \le m_0. \tag{5.36}$$

Note first that $\overline{p} = i(c^T) + p(c)$ is always even by (B) of Theorem 4.3. Therefore by (5.35) we get

$$\epsilon(c^{T+b}) = (-1)^{i(c^{T+b})-i(c)} = (-1)^{i(c^b)-i(c)} = \epsilon(c^b). \tag{5.37}$$

For any $0 \le a < b \le m_0$, since $\nu(c^{T+b}) = \nu(c^b)$ holds in (5.35) and $b|(T+b)$ holds in (5.36), it follows from Lemma 2.1 and (iii) of Lemma 2.2 that

$$\begin{aligned}
H_h(\overline{\Lambda}^{\kappa_b}, \overline{\Lambda}^{\kappa_b\#}) &= \overline{C}_h(E, c^b) \\
&= H_{h-i(c^b)}(N^-_{c^b} \cup \{c^b\}, N^-_{c^b})^{\epsilon(c^b)\mathbf{Z}_b} \\
&= H_{h-i(c^b)}(N^-_{c^{T+b}} \cup \{c^{T+b}\}, N^-_{c^{T+b}})^{\epsilon(c^{T+b})\mathbf{Z}_{T+b}} \\
&= H_{h+\overline{p}-i(c^{T+b})}(N^-_{c^{T+b}} \cup \{c^{T+b}\}, N^-_{c^{T+b}})^{\epsilon(c^{T+b})\mathbf{Z}_{T+b}} \\
&= \overline{C}_{h+\overline{p}}(E, c^{T+b}) \\
&= H_{h+\overline{p}}(\overline{\Lambda}^{\kappa_{T+b}}, \overline{\Lambda}^{\kappa_{T+b}\#}).
\end{aligned} \tag{5.38}$$

Here we used Lemma 2.1 in the second and fifth equalities, (iii) of Lemma 2.2 and (5.35)-(5.36) in the third one and (5.35) in the fourth one.

The case of $b - a = 1$ is proved.

Step 2. *The induction argument for general $b > a$.*

Now we can follow precisely the proof in the Step 2 on pages 1789-1792 of Theorem 4.3 of [LoD1] and complete the proof of Theorem 5.3 here. Thus we omit all these details here. ∎

Next we generalize the Proposition 5.1 of [LoD1] for rational closed geodesics to irrational ones. Here we denote by $\mathbf{Q}^m$ the m times of the module instead of using the notation $m\mathbf{Q}$ in order to make the text clearer.

Theorem 5.4. *Let c be the only one prime closed geodesic on a compact Finsler manifold (M, F) of dimension d. Suppose c is orientable. Let $n = n(c)$ be the analytical period of c and $m_0 = m_0(c)$ be given by Definition 4.1. Let $T \in (m_0!n)\mathbf{N}$ be the even integer given by Theorems 4.3-4.5 and 5.3. Denote by $X_j = H_j(\overline{\Lambda}, \overline{\Lambda}^T) = \mathbf{Q}^{x_j}$ for all $j \in \mathbf{Z}$. Then there holds*

$$x_j = b_{j-i(c^T)-p(c)} \qquad \forall\, 0 \le j \le i(c^T) + p(c) + d - 2, \tag{5.39}$$

where b_j's are the Betti numbers of the loop space ΛM defined in Section 2.

Proof. Let $R = i(c^T)$. Firstly we fix an integer $j \le R + p(c) + d - 2$. Because there is only one prime closed geodesic c on M, there holds $\hat{i}(c) > 0$. Thus we have $i(c^m) \to +\infty$ as $m \to +\infty$. According to Definition 4.1 and Lemma 4.4 we have

$$i(c^m) \ge R + p(c) + d, \qquad \forall\, m \ge T + m_0.$$

It then implies

$$\overline{C}_q(E, c^m) = 0, \qquad \forall\, m \ge T + m_0, \quad q \le j + 1 = R + p(c) + d - 1.$$

Therefore by Theorem II.1.5 on page 89 of [Cha1] we obtain

$$H_q(\overline{\Lambda}, \overline{\Lambda}^{\kappa_m}) = 0 \qquad \forall\, m \ge T + m_0, \;\; 0 \le q \le j + 1. \tag{5.40}$$

Thus the exact sequence of the triple $(\overline{\Lambda}, \overline{\Lambda}^{\kappa_{m_0+T}}, \overline{\Lambda}^{\kappa_T})$ yields

$$0 = H_{j+1}(\overline{\Lambda}, \overline{\Lambda}^{\kappa_{m_0+T}}) \to H_j(\overline{\Lambda}^{\kappa_{m_0+T}}, \overline{\Lambda}^{\kappa_T}) \to H_j(\overline{\Lambda}, \overline{\Lambda}^{\kappa_T}) \to H_j(\overline{\Lambda}, \overline{\Lambda}^{\kappa_{m_0+T}}) = 0, \tag{5.41}$$

which then implies the isomorphism:

$$H_j(\overline{\Lambda}^{\kappa_{m_0+T}}, \overline{\Lambda}^{\kappa_T}) = H_j(\overline{\Lambda}, \overline{\Lambda}^{\kappa_T}) = \mathbf{Q}^{x_j}. \tag{5.42}$$

On the other hand, by Theorem 5.3 we obtain an isomorphism:

$$H_{j-R-p(c)}(\overline{\Lambda}^{\kappa_{m_0}}, \overline{\Lambda}^0) = H_j(\overline{\Lambda}^{\kappa_{m_0+T}}, \overline{\Lambda}^{\kappa_T}), \quad \forall\, j \le R + p(c) + d - 2. \tag{5.43}$$

Fix an integer $l \le d - 2$. By Definition 4.1 we have $i(c^m) \ge d + 4k$ for all $m \ge m_0$, which implies

$$H_q(\overline{\Lambda}, \overline{\Lambda}^{\kappa_m}) = 0 \qquad \forall\, m \ge m_0, \;\; 0 \le q \le l + 1. \tag{5.44}$$

Then the exact sequence of the triple $(\overline{\Lambda}, \overline{\Lambda}^{\kappa_{m_0}}, \overline{\Lambda}^0)$ yields

$$\begin{aligned} 0 \;=\; & H_{j-R-p(c)+1}(\overline{\Lambda}, \overline{\Lambda}^{\kappa_{m_0}}) \to H_{j-R-p(c)}(\overline{\Lambda}^{\kappa_{m_0}}, \overline{\Lambda}^0) \\ & \to H_{j-R-p(c)}(\overline{\Lambda}, \overline{\Lambda}^0) \to H_{j-R-p(c)}(\overline{\Lambda}, \overline{\Lambda}^{\kappa_{m_0}}) = 0, \quad \forall\, j \le R + p(c) + d - 2. \end{aligned}$$

It then implies the isomorphism:

$$H_{j-R-p(c)}(\overline{\Lambda}^{\kappa_{m_0}}, \overline{\Lambda}^0) = H_{j-R-p(c)}(\overline{\Lambda}, \overline{\Lambda}^0) = \mathbf{Q}^{b_{j-R-p(c)}}, \quad \forall\, j \le R + p(c) + d - 2. \tag{5.45}$$

Therefore (5.42)-(5.43) and (5.45) yield the claim (5.39). ∎

Next we generalize the Theorems 5.2 of [LoD1] for the rational closed geodesics to irrational ones.

Theorem 5.5. *Let (M,F) be a compact simply connected dh-dimensional Finsler manifold with $H^*(M,\mathbf{Q}) = T_{d,h+1}(x)$ for some integers $d \geq 2$ and $h \geq 1$. Suppose c is the only one prime closed geodesic on M, and let $\mu = p(c) + dh - 3$. Denote by $n = n(c)$ and $m_0 = m_0(c)$ given by (4.1) and Definition 4.1. Then there exist an even integer $T \in (m_0!n)\mathbf{N}$ and an integer $\kappa \geq 0$ such that*

$$B(d,h)(i(c^T) + p(c)) + (-1)^{\mu+i(c^T)}\kappa = \sum_{j=\mu-p(c)+1}^{i(c^T)+\mu} (-1)^j b_j, \tag{5.46}$$

where $B(d,h)$ is given in Lemma 2.4.

Proof. Note first that by Proposition 3.5 and Remark 3.6 the closed geodesic c on M is orientable because M is simply connected.

Since there exists only one prime closed geodesic c, it follows that $\hat{i}(c) > 0$ and $0 \leq i(c) \leq d-1$. Specially by Lemma 2.4 we obtain

$$\hat{i}(c) \in \mathbf{Q}. \tag{5.47}$$

Let

$$d_j = k_j^{\epsilon(c^n)}(c^n), \qquad \forall j \in \mathbf{Z}. \tag{5.48}$$

Then by the definition of $n = n(c)$, Lemma 2.3 and (4.2) we obtain

$$k_j^{\epsilon(c^{mn})}(c^{mn}) = d_j, \qquad \forall j \in \mathbf{Z}, \quad m \in \mathbf{N}. \tag{5.49}$$

Fix $T \in (m_0!n)\mathbf{N}$ to be an even integer determined by Theorems 4.3-4.5, 5.3, and 5.5. Specially we require that this T makes (4.12) hold.

Then we claim the following four conditions hold:

$$i(c^{m+T}) = i(c^T) + i(c^m) + p(c), \qquad \forall\, 1 \leq m \leq m_0, \tag{5.50}$$

$$i(c^m) + \nu(c^m) \leq i(c^T) + \mu, \qquad \forall\, 1 \leq m < T, \tag{5.51}$$

$$d_j = 0, \qquad \forall\, j \geq \mu + 2, \tag{5.52}$$

$$H_{i(c^T)+\mu+1}(\overline{\Lambda}, \overline{\Lambda}^{\kappa_T}) = 0. \tag{5.53}$$

In fact, (5.50) follows from (A) of Theorem 4.3, and (5.51) follows from Theorem 4.5.

Note that if $k \geq 1$ in Theorem 4.3, there holds $A \geq 1$ by Proposition 3.7. Thus for $j \geq \mu + 2 = p(c) + dh - 1$, it yields $j > \nu(c^n)$ by (C) of Theorem 4.3, which implies that (5.52) holds. If $k = 0$, then (5.52) was proved in the proof of Theorem 6.1 of [LoD1] when verifying the condition (5.11) there via Hingston's Theorem of [Hin2] (cf. Theorem 4.1 of [LoD1]).

Note that $i(c^T) + \mu + 1 = i(c^T) + p(c) + dh - 2$ holds. So by Theorem 5.4, Lemmas 2.5 and 2.6, we obtain

$$H_{i(c^T)+\mu+1}(\overline{\Lambda}, \overline{\Lambda}^{\kappa_T}) = b_{dh-2} = 0.$$

Thus (5.53) holds, and the proof of the four conditions (5.50)-(5.53) is complete.

Let $R = i(c^T)$. Then by (5.50)-(5.53) and Lemma 4.4 we obtain the following distribution diagram (5.54) of $\dim \overline{C}_j(E, c^m)$ for any $j \geq 0$ and $m \geq 1$.

$$
\begin{array}{c|ccccccccccccc}
\cdots & & & & & & & & & & & & * & \cdots \\
T+m_0+1 & & & & & & & & & & & & * & \cdots \\
T+m_0 & & & & & & * & \cdots & \cdots & \cdots & \cdots & * & \cdots \\
\cdots & & & & & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
T+1 & & & & & & * & \cdots & \cdots & \cdots & \cdots & * & \cdots \\
T & & & 0 & d_0 & \cdots & d_{p(c)} & \cdots & d_\mu & d_{\mu+1} & d_{\mu+2} & 0 & 0 \\
T-1 & * & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & * & & & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & & & \\
1 & * & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & * & & & & \\
\hline
m \text{ in } c^m & c_0 & \cdots & c_{R-1} & c_R & \cdots & c_{R+p(c)} & \cdots & c_{R+\mu} & c_{R+\mu+1} & c_{R+\mu+2} & c_{R+\mu+3} & \cdots
\end{array} \tag{5.54}
$$

Here as coordinates of the diagram (5.54), the first left column lists the iteration time m of c^m starting from 1 to $T + m_0 + 1$ and upwards, and the first row from below lists the dimensions $c_j = \dim \overline{C}_j(E, c^*)$ of the S^1-equivariant critical module $\overline{C}_j = \overline{C}_j(E, c^*)$ from $j = 0$ to $j = R+\mu+3$ and rightwards. The entry $D_j(c^m)$ in this diagram at m-th row and j-th column is given by $D_j(c^m) = \dim \overline{C}_j(E, c^m)$. Here $d_j = \dim \overline{C}_j(E, c^T)$s are shown in this diagram. $*$s and Dots in this diagram indicate entries which may not be zero whose precise values depend on $\dim \overline{C}_j(E, c^m)$. Entries on the empty places in the diagram are all 0.

Now the proof is similar to that of Theorem 5.2 in [LoD1] (cf. pages 1795-1799 for more details). Here for reader's conveniences, we include certain details of the proof here.

Denote by $\kappa_m = E(c^m)$ for $m \geq 1$. As in the Step 1 of the proof of Theorem 5.2 of [LoD1], for $j \in \mathbf{Z}$, we denote by

$$
U_j = H_j(\overline{\Lambda}^{\kappa_T}, \overline{\Lambda}^0) = \mathbf{Q}^{u_j}, \quad B_j = H_j(\overline{\Lambda}, \overline{\Lambda}^0) = \mathbf{Q}^{b_j}, \quad X_j = H_j(\overline{\Lambda}, \overline{\Lambda}^{\kappa_T}) = \mathbf{Q}^{x_j}. \tag{5.55}
$$

Then the long exact sequence of the triple $(\overline{\Lambda}, \overline{\Lambda}^{\kappa_T}, \overline{\Lambda}^0)$ yields the following diagram:

$$
\begin{array}{ccccccccccccccc}
X_{R+\mu+1} & \to & U_{R+\mu} & \to & B_{R+\mu} & \to & X_{R+\mu} & \to & \cdots & \to & U_0 & \to & B_0 & \to & X_0 \\
\| & & \| & & \| & & \| & & & & \| & & \| & & \| \\
0 & & \mathbf{Q}^{u_{R+\mu}} & & \mathbf{Q}^{b_{R+\mu}} & & \mathbf{Q}^{x_{R+\mu}} & & \cdots & & \mathbf{Q}^{u_0} & & 0 & & 0,
\end{array}
$$

where $X_{R+\mu+1} = 0 = X_0$ follows from (5.53), Theorem 5.4 and Lemmas 2.5 and 2.6. $B_0 = 0$ follows from Lemmas 2.5 and 2.6. Then this long exact sequence yields

$$
0 = \sum_{j=0}^{R+\mu} (-1)^j (u_j - b_j + x_j). \tag{5.56}
$$

Because $T \geq 2$, for $j \in \mathbf{Z}$ besides U_j defined in (5.55) we denote by

$$
V_j = H_j(\overline{\Lambda}^{\kappa_{T-1}}, \overline{\Lambda}^0) = \mathbf{Q}^{v_j}, \quad E_j = H_j(\overline{\Lambda}^{\kappa_T}, \overline{\Lambda}^{\kappa_{T-1}}) = \mathbf{Q}^{e_j}.
$$

Then the exact sequence of the triple $(\overline{\Lambda}^{\kappa_T}, \overline{\Lambda}^{\kappa_{T-1}}, \overline{\Lambda}^0)$ and the diagram (5.54) yield the following diagram:

$$\begin{array}{ccccccccc}
V_{R+\mu+1} & \to & U_{R+\mu+1} & \to & E_{R+\mu+1} & \to & V_{R+\mu} & \to & \cdots \\
\| & & \| & & \| & & \| & & \\
0 & & \mathbf{Q}^{u_{R+\mu+1}} & & \mathbf{Q}^{e_{R+\mu+1}} & & \mathbf{Q}^{v_{R+\mu}} & & \cdots \\
 & & & & & & & & \\
 & \to & V_R & \to & U_R & \to & E_R & \to & \cdots \\
 & & \| & & \| & & \| & & \\
 & & \mathbf{Q}^{v_R} & & \mathbf{Q}^{u_R} & & \mathbf{Q}^{e_R} & & \cdots \\
 & & & & & & & & \\
 & \to & V_0 & \to & U_0 & \to & E_0 & \to & 0 \\
 & & \| & & \| & & \| & & \\
 & & \mathbf{Q}^{v_0} & & \mathbf{Q}^{u_0} & & \mathbf{Q}^{e_0} & &
\end{array}$$

where $V_{R+\mu+1} = 0$ follows from (5.51) and the diagram (5.54). Then this long exact sequence yields

$$\sum_{j=0}^{R+\mu}(-1)^j u_j = (-1)^{R+\mu} u_{R+\mu+1} + \sum_{j=0}^{R+\mu}(-1)^j v_j + \sum_{j=0}^{R+\mu+1}(-1)^j e_j. \tag{5.57}$$

Note that by (5.49) we have

$$e_j = \begin{cases} d_{j-R}, & \text{for } R \le j \le R+\mu+1, \\ 0, & \text{otherwise.} \end{cases} \tag{5.58}$$

Thus we obtain

$$\sum_{j=0}^{R+\mu}(-1)^j u_j = (-1)^{R+\mu} u_{R+\mu+1} + \sum_{j=0}^{R+\mu}(-1)^j v_j + \sum_{j=0}^{\mu+1}(-1)^{R+j} d_j. \tag{5.59}$$

Now combining (5.56) and (5.59) we obtain

$$0 = \sum_{j=0}^{R+\mu}(-1)^j v_j + \sum_{j=0}^{\mu+1}(-1)^{R+j} d_j - \sum_{j=0}^{R+\mu}(-1)^j b_j + \sum_{j=0}^{R+\mu}(-1)^j x_j + (-1)^{R+\mu} u_{R+\mu+1}. \tag{5.60}$$

Now as in [LoD1], we can apply the procedure above to decrease the level sets one by one by induction. In this way, each time we pass through a critical level $E(c^m)$ with $m \le T$, the term $\sum_{j=0}^{R+\mu}(-1)^j v_j$ on the right hand side of (5.60) will be replaced by the sum of a similar alternating sum of dimensions of homological modules of a new lower level set pair $(\overline{\Lambda}^{\kappa_{m-1}}, \overline{\Lambda}^0)$ and a term $\sum_{j=0}^{\nu(c^m)}(-1)^{i(c^m)+j} k_j^{\epsilon(c^m)}(c^m)$. Here the sign of $i(c^m)$ indicates the parity of the number of column in which the term $k_0^{\epsilon(c^m)}(c^m)$ appears. Then by induction from (5.56)-(5.60) repeating the proof of Theorem 5.2 in [LoD1] by using the above diagram (5.54) and our Theorem 5.4, similarly to (5.22)

of [LoD1] we obtain

$$
\begin{aligned}
0 \;=\;& \frac{T}{n}\sum_{j=0}^{\mu+1}(-1)^{i(c^n)+j}d_j + \frac{T}{n}\sum_{m=1}^{n-1}\sum_{j=0}^{\nu(c^m)}(-1)^{i(c^m)+j}k_j^{\epsilon(c^m)}(c^m) \\
& - \sum_{j=0}^{R+\mu}(-1)^j b_j + \sum_{j=0}^{R+\mu}(-1)^j b_{j-R-p(c)} + (-1)^{R+\mu}u_{R+\mu+1}. \qquad (5.61)
\end{aligned}
$$

Note that in the proof of (5.61), the facts that T is an integer multiple of $n(c)$, the $n(c)$-periodicity of critical modules in iterates given by Lemma 2.3, (4.2) and (5.49) are crucial.

Now similarly to (5.23) of [LoD1] we can apply the mean index identity Lemma 2.4 to further obtain

$$
\begin{aligned}
B(d,h)n\hat{i}(c) &= \sum_{\substack{1\le m\le n\\ 0\le j\le 2dh-2}}(-1)^{i(c^m)+j}k_j^{\epsilon_j}(c^m) \\
&= \sum_{\substack{1\le m\le n-1\\ 0\le j\le 2dh-2}}(-1)^{i(c^m)+j}k_j^{\epsilon_j}(c^m) + \sum_{j=0}^{\nu(c^n)}(-1)^{i(c^n)+j}k_j^{\epsilon_n}(c^n) \\
&= \sum_{m=1}^{n-1}\sum_{j=0}^{\nu(c^m)}(-1)^{i(c^m)+j}k_j^{\epsilon_j}(c^m) + \sum_{j=0}^{\nu(c^n)}(-1)^{i(c^n)+j}k_j^{\epsilon_n}(c^n) \\
&= \sum_{m=1}^{n-1}\sum_{j=0}^{\nu(c^m)}(-1)^{i(c^m)+j}k_j^{\epsilon_j}(c^m) \\
&\qquad + \sum_{j=0}^{\mu+1}(-1)^{i(c^n)+j}d_j + \sum_{j=\mu+2}^{\nu(c^n)}(-1)^{i(c^n)+j}d_j \\
&= \sum_{m=1}^{n-1}\sum_{j=0}^{\nu(c^m)}(-1)^{i(c^m)+j}k_j^{\epsilon_j}(c^m) + \sum_{j=0}^{\mu+1}(-1)^{i(c^n)+j}d_j, \qquad (5.62)
\end{aligned}
$$

where we have used the condition $d_j = 0$ for all $j \ge \mu+2$ of (5.52) in the last equality.

Now by (D) of Theorem 4.3, the rationality (5.47) of $\hat{i}(c)$, (5.61) and (5.62) we obtain

$$
\begin{aligned}
0 &= B(d,h)T\hat{i}(c) - \sum_{j=0}^{R+\mu}(-1)^j b_j + \sum_{j=0}^{R+\mu}(-1)^j b_{j-R-p(c)} + (-1)^{R+\mu}u_{R+\mu+1} \\
&= B(d,h)(R+p(c)) - \sum_{j=0}^{R+\mu}(-1)^j b_j + \sum_{j=0}^{R+\mu}(-1)^j b_{j-R-p(c)} + (-1)^{R+\mu}u_{R+\mu+1} \\
&= B(d,h)(R+p(c)) - \sum_{j=\mu-p(c)+1}^{R+\mu}(-1)^j b_j + (-1)^{R+\mu}u_{R+\mu+1}. \qquad (5.63)
\end{aligned}
$$

That is, (5.46) holds with $\kappa = u_{R+\mu+1} \ge 0$.

This completes the proof of Theorem 5.5. ∎

6 Proofs of Theorems 1.1 and 1.2

In this section, we will follow ideas from Section 6 of [LoD1] and Section 4 of [DuL3] to give the proofs of Theorems 1.1 and 1.2 via replacing $n = n(c)$ by the integer T obtained by Theorems 4.3, 4.5, 5.3 and 5.5, and modifying related arguments using our above results. For reader's conveniences and completeness, we give all the details here.

Proof of Theorem 1.1. Let M be a compact simply connected manifold of dimension not less than 2 with a Finsler metric F. By Theorems A and B in the Section 1, it suffices to assume that the condition (1.3) on M holds, i.e.,

$$H^*(M;\mathbf{Q}) \cong T_{d,h+1}(x) = \mathbf{Q}[x]/(x^{h+1} = 0)$$

with a generator x of degree $d \geq 2$ and hight $h + 1 \geq 2$.

We prove the theorem by contradiction. Thus we assume that there exists only one prime closed geodesic c on (M, F). To generate the non-trivial $H_{d-1}(\Lambda M/S^1, \Lambda M^0/S^1;\mathbf{Q})$ (cf. Lemmas 2.5 and 2.6), this c must satisfy

$$0 \leq i(c) \leq d - 1, \quad \hat{i}(c) > 0, \quad \hat{i}(c) \in \mathbf{Q}. \tag{6.1}$$

where the last conclusion follows from Lemma 2.4.

For the analytic period $n = n(c)$ and $m_0 = m_0(c)$ given by Definition 4.1, fix a large even integer $T \in (m_0!n)\mathbf{N}$ determined by Theorems 4.3, 4.5, 5.3 and 5.5. Then by (6.1) and (D) of Theorem 4.3 we have

$$i(c^T) + p(c) = T\hat{i}(c) > 0.$$

Note that $i(c^T) = p(c) \pmod 2$ by (B) of Theorem 4.3, so we obtain

$$i(c^T) + p(c) \in 2\mathbf{N}. \tag{6.2}$$

Let $\mu = p(c) + (dh - 3)$. Then by (6.2) we have

$$i(c^T) + \mu \geq dh - 1 \geq 1, \qquad i(c^T) + \mu \in 2\mathbf{N}_0 + (dh - 1). \tag{6.3}$$

Then by Theorem 5.5, we obtain for some integer $\kappa \geq 0$:

$$B(d, h)(i(c^T) + p(c)) + (-1)^{i(c^T)+\mu}\kappa = \sum_{j=\mu-p(c)+1}^{i(c^T)+\mu} (-1)^j b_j. \tag{6.4}$$

Note that when d is odd, then $h = 1$ by Remark 2.5 of [Rad1]. And when $h = 1$, M is rationally homotopic to the sphere S^d. So we can classify the manifolds M satisfying (1.3) into two classes according to the parity of d, and continue our proof correspondingly.

Case 1. $d \geq 2$ is even and $h \geq 1$.

Note that, in this case, $i(c^T)+\mu$ is odd by (6.3). And there holds $b_{2j}=0$ for all $j\in\mathbf{N}_0$ by Lemma 2.6. Thus by (6.4) we obtain

$$B(d,h)(i(c^T)+p(c)) \ge -\sum_{2j-1=\mu-p(c)+1}^{i(c^T)+\mu} b_{2j-1}. \tag{6.5}$$

Let $D=d(h+1)-2$. By Lemma 2.4 we have

$$B(d,h)=-\frac{h(h+1)d}{2D}<0.$$

Thus from (B) of Theorem 4.3, we have

$$i(c^T)+\mu-(d-1)=i(c^T)+p(c)+dh-d-2\in 2\mathbf{N}. \tag{6.6}$$

By (6.5), (6.6) and (2.12) we obtain

$$\begin{aligned} i(c^T)+p(c) &\le -\frac{1}{B(d,h)}\sum_{2j-1=\mu-p(c)+1}^{i(c^T)+\mu} b_{2j-1} \\ &= \frac{2D}{h(h+1)d}\left(\sum_{2j-1=1}^{i(c^T)+\mu} b_{2j-1}-\sum_{2j-1=1}^{dh-2} b_{2j-1}\right). \end{aligned} \tag{6.7}$$

Note that here because $i(c^T)+p(c)\ge 2$ by (6.2), we have

$$i(c^T)+\mu=i(c^T)+p(c)+dh-3\ge dh-1=d-1+(h-1)d. \tag{6.8}$$

By Lemma 2.6 we have

$$\sum_{2j-1=1}^{i(c^T)+\mu} b_{2j-1}=\frac{h(h+1)d}{2D}\Big(i(c^T)+\mu-(d-1)\Big)-\frac{h(h-1)d}{4}+1+\epsilon_{d,h}(i(c^T)+\mu). \tag{6.9}$$

On the other hand, because $dh-3<dh-1=d-1+(h-1)d$, by Lemma 2.6 we have

$$\begin{aligned} \sum_{0\le 2j-1\le dh-3} b_{2j-1} &= \sum_{d-1\le 2j-1\le dh-3}\left([\frac{2j-1-(d-1)}{d}]+1\right) \\ &= \sum_{d\le 2j\le dh-2}[\frac{2j}{d}] \\ &= \sum_{\frac{d}{2}\le j\le \frac{dh}{2}-1}[\frac{j}{d/2}] \\ &= \sum_{i=1}^{h-1}\sum_{j=\frac{id}{2}}^{\frac{(i+1)d}{2}-1}[\frac{j}{d/2}] \\ &= \frac{d}{2}\sum_{i=1}^{h-1} i \\ &= \frac{dh(h-1)}{4}. \end{aligned} \tag{6.10}$$

Therefore we get

$$
\begin{aligned}
&\sum_{0\le 2j-1\le i(c^T)+\mu} b_{2j-1} - \sum_{0\le 2j-1\le dh-3} b_{2j-1} \\
&\quad = \frac{h(h+1)d}{2D}\Big(i(c^T)+\mu-(d-1)\Big) - \frac{h(h-1)d}{4} + 1 + \epsilon_{d,h}(i(c^T)+\mu) - \frac{dh(h-1)}{4} \\
&\quad = \frac{h(h+1)d}{2D}\Big(i(c^T)+p(c)+dh-d-2\Big) - \frac{dh(h-1)}{2} + 1 + \epsilon_{d,h}(i(c^T)+\mu). \qquad (6.11)
\end{aligned}
$$

Then (6.7) becomes

$$
i(c^T)+p(c) \le i(c^T)+p(c)+dh-d-2+\frac{2D}{h(h+1)d}\left(1-\frac{dh(h-1)}{2}+\epsilon_{d,h}(i(c^T)+\mu)\right),
$$

that is,

$$
\begin{aligned}
\epsilon_{d,h}(i(c^T)+\mu) \quad &\ge \quad \frac{h(h+1)d}{2D}\left(d+2+\frac{(h-1)D}{h+1}-dh-\frac{2D}{h(h+1)d}\right) \\
&= \quad \frac{dh-(d-2)}{dh+(d-2)}. \qquad (6.12)
\end{aligned}
$$

Note that by (6.6) we have

$$
i(c^T)+\mu-(d-1) = i(c^T)+p(c)+dh-d-2 = i(c^T)+p(c)-2d+D. \qquad (6.13)
$$

Let $\eta\in[0,D/2-1]$ be an integer such that

$$
\frac{2\eta}{D} = \{\frac{i(c^T)+p(c)-2d}{D}\} = \{\frac{i(c^T)+\mu-(d-1)}{D}\}. \qquad (6.14)
$$

By the definition (2.13) of $\epsilon_{d,h}(i(c^T)+\mu)$ and (6.14), we obtain

$$
\begin{aligned}
\epsilon_{d,h}(i(c^T)+\mu) \quad &= \quad \{\frac{D}{dh}\{\frac{i(c^T)+\mu-(d-1)}{D}\}\} - (\frac{2}{d}+\frac{d-2}{dh})\{\frac{i(c^T)+\mu-(d-1)}{D}\} \\
&\qquad -h\{\frac{D}{2}\{\frac{i(c^T)+\mu-(d-1)}{D}\}\} - \{\frac{D}{d}\{\frac{i(c^T)+\mu-(d-1)}{D}\}\} \\
&= \quad \{\frac{2\eta}{dh}\} - (\frac{2}{d}+\frac{d-2}{dh})\frac{2\eta}{D} - h\{\frac{2\eta}{2}\} - \{\frac{2\eta}{d}\} \\
&= \quad \{\frac{2\eta}{dh}\} - (\frac{2}{d}+\frac{d-2}{dh})\frac{2\eta}{D} - \{\frac{2\eta}{d}\} \\
&\equiv \quad \epsilon(2\eta). \qquad (6.15)
\end{aligned}
$$

Now we claim

$$
\epsilon(2\eta) < \frac{dh-(d-2)}{dh+(d-2)}, \qquad \forall\, 2\eta\in[0,dh-2]. \qquad (6.16)
$$

In fact, we write

$$
2\eta = pd+2m \qquad \text{with some } \ p\in\mathbf{N}_0,\ \ 2m\in[0,d-2]. \qquad (6.17)
$$

Then from $pd + 2m = 2\eta \le dh - 2 = (h-1)d + d - 2$ we have

$$p \in [0, h-1]. \tag{6.18}$$

Therefore in this case we obtain

$$\begin{aligned}
\epsilon(2\eta) &= \frac{pd+2m}{dh} - (\frac{2}{d} + \frac{d-2}{dh})\frac{pd+2m}{D} - \frac{2m}{d} \\
&= \frac{p}{h} - \frac{(2h+d-2)p}{hD} + \frac{2m}{dh} - \frac{(2h+d-2)2m}{dhD} - \frac{2m}{d} \\
&= \frac{p}{h}(1 - \frac{2h+d-2}{D}) + \frac{2m}{d}(\frac{1}{h} - \frac{2h+d-2}{hD} - 1) \\
&= \frac{p(d-2) - 2mh}{D} \\
&\le \frac{(h-1)(d-2)}{D}.
\end{aligned} \tag{6.19}$$

Now if (6.16) does not hold, we then obtain

$$\frac{dh-(d-2)}{D} \le \epsilon(2\eta) \le \frac{(h-1)(d-2)}{D},$$

that is,

$$dh - d + 2 \le dh - d + 2 - 2h.$$

Because $h \ge 1$, this yields a contradiction and completes the proof of (6.16).

If $d = 2$, there holds $D - 2 = dh + d - 4 = dh - 2$. Thus (6.16) holds for any integer $2\eta \in [0, D-2]$.

If $d \ge 4$, for any $2\eta \in [dh, D-2]$, write $2\eta = pdh + 2m$ for some $p \in \mathbf{N}_0$ and $2m \in [0, dh-2]$. Then from $D - 2 = (h+1)d - 4 = hd + d - 4$ we obtain $p \le 1$ and $2m \le d - 4$. Thus we have

$$\begin{aligned}
\epsilon(2\eta) &= \frac{2m}{dh} - (\frac{2}{d} + \frac{d-2}{dh})\frac{pdh+2m}{D} - \frac{2m}{d} \\
&= \epsilon(2m) - (\frac{2}{d} + \frac{d-2}{dh})\frac{pdh}{D} \\
&\le \epsilon(2m).
\end{aligned} \tag{6.20}$$

Therefore from (6.16) and (6.20), it yields

$$\epsilon(2\eta) < \frac{dh-(d-2)}{dh+(d-2)}, \qquad \forall\, \eta \in [0, D/2 - 1]. \tag{6.21}$$

Together with (6.12), (6.15), and the choice (6.14) of 2η, we then obtain

$$\frac{dh-(d-2)}{dh+(d-2)} \le \epsilon_{d,h}(i(c^T) + \mu) = \epsilon(2\eta) < \frac{dh-(d-2)}{dh+(d-2)}. \tag{6.22}$$

This contradiction completes the proof of Case 1.

Case 2. *$d \ge 2$ is odd and $h = 1$.*

In this case, M is rationally homotopic to the sphere S^d. Note that $i(c^T)+\mu$ is even by (6.3). Because $\mu-p(c)+1=d-2$, there holds $b_j=0$ for any $j\le\mu-p(c)+1$ by Lemma 2.5. Thus by Theorem 5.5 and Lemma 2.5 we obtain

$$\begin{aligned} i(c^T)+p(c) &\le \frac{1}{B(d,1)}\sum_{2j=0}^{i(c^T)+\mu} b_{2j} \\ &\le i(c^T)+p(c)+d-3-\frac{d-1}{2}\frac{2(d-1)}{d+1} \\ &= i(c^T)+p(c)-\frac{4}{d+1}. \end{aligned} \tag{6.23}$$

This is a contradiction.

Now the proof of Theorem 1.1 is complete. ∎

Now we give

Proof of Theorem 1.2. Here arguments are the same as in Section 7 of [LoD1]. For any reversible Finsler as well as Riemannian metric F on a compact manifold M, the energy functional E is symmetric on every loop $f\in\Lambda M$ and its inverse curve f^{-1} defined by $f^{-1}(t)=f(1-t)$. Thus these two curves have the same energy $E(f)=E(f^{-1})$ and play the same roles in the variational structure of the energy functional E on ΛM. Specially, the m-th iterates c^m and c^{-m} of a prime closed geodesic c and its inverse curve c^{-1} have precisely the same Morse indices, nullities, and critical modules. Let $n=n(c)=n(c^{-1})$. Then there holds

$$\dim\overline{C}_*(E,c^m)=\dim\overline{C}_*(E,c^{-m}). \tag{6.24}$$

Thus if c is the only geometrically distinct prime closed geodesic on M, each entry in the diagram (5.54) in the reversible case should be doubled. So (5.46) in Theorem 5.5 becomes

$$B(d,h)(i(c^T)+p(c))+(-1)^{\mu+i(c^T)}2\kappa=\sum_{j=\mu-p(c)+1}^{\mu+i(c^T)}(-1)^j b_j. \tag{6.25}$$

These changes bring no influence to our proofs in Section 6. Therefore our above proof yields two geometrically distinct closed geodesics for reversible Finsler metrics too. ∎

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